

# Affine connections and singularities of tangent surfaces to space curves

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## 1 Introduction

The tangent lines to a space curve form a ruled surface, which is called the *tangent surface* or the *tangent developable* or the *tangential variety* to the space curve (see for instance [26][6][19][28]). Tangent surfaces appear in various geometric problems and applications naturally, providing several important examples of non-isolated singularities in applications of geometry (see for instance [1][7][10][21][22][23][31][32]).

It is known, in the three dimensional Euclidean space  $E^3$ , that the tangent surface to a generic space curve  $\gamma : I \rightarrow E^3$  is locally diffeomorphic to the *cuspidal edge* or to the *folded umbrella* (also called, *cuspidal cross cap*), as is found by Cayley and Cleave [8]. Cuspidal edge singularities appear along ordinary points where  $\gamma', \gamma'', \gamma'''$  are linearly independent, while the folded umbrella appears at an isolated point of zero torsion where  $\gamma', \gamma'', \gamma'''$  are linearly dependent but  $\gamma', \gamma'', \gamma''''$  are linearly independent (see [4][33]).

The classification is generalized to more degenerate cases by Mond [29][30] and Shcherbak [37][38][3]. See also [18][19]. The classifications were performed mainly in locally projectively flat cases so far. However more general cases, namely, not necessarily projectively flat cases have never been treated as far as the authors recognize.

In this paper we give the complete solution to the local diffeomorphism classification problem of generic singularities which appear in tangent surfaces, in as wider situations as possible. We interpret *geodesics* as “lines” whenever a (semi-)Riemannian metric, or, more generally, an affine connection  $\nabla$  is given in an ambient space of arbitrary dimension. Then, given an immersed curve, or, more generally a *directed* curve or a *frontal* curve which has well-defined tangent directions along the curve, we define  $\nabla$ -*tangent surface* by the ruled surface by tangent geodesics to the curve. Then the main theorems in this paper are as follows:

**Theorem 1.1** (Genericity 1: Singularities of tangent surfaces to generic curves) *Let  $\nabla$  be any affine connection on a manifold  $M$  of dimension  $m \geq 3$ . The singularities of the  $\nabla$ -tangent surface to a generic curve in  $M$  on a neighborhood of the curve are only the cuspidal edges and the folded umbrellas if  $m = 3$ , and the embedded cuspidal edges if  $m \geq 4$ .*

**Theorem 1.2** (Genericity 2: Singularities of tangent surfaces to generic directed curves) *Let  $\nabla$  be any affine connection on a manifold  $M$  of dimension  $m \geq 3$ . The singularities of the  $\nabla$ -tangent surface to a generic directed curve in  $M$  on a neighborhood of the curve are only the cuspidal edges, the folded umbrellas and the swallowtails if  $m = 3$ , and the embedded cuspidal edges and open swallowtails if  $m \geq 4$ .*

The genericity is exactly given in each case of genericity 1 and 2 (see Propositions 10.1, 11.1) using Whitney  $C^\infty$  topology on appropriate space of curves.

**Theorem 1.3** (Characterization) *Let  $\nabla$  be a torsion free affine connection on a manifold  $M$ . Let  $\gamma : I \rightarrow M$  be a  $C^\infty$  curve from an open interval  $I$ .*

(1) *Let  $\dim(M) = 3$ . If  $(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), (\nabla^3\gamma)(t_0)$  are linearly independent, then the  $\nabla$ -tangent surface  $\nabla\text{-Tan}(\gamma)$  is locally diffeomorphic to the cuspidal edge at  $(t_0, 0) \in I \times \mathbf{R}$ . If  $(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), (\nabla^3\gamma)(t_0)$  are linearly dependent, and  $(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), (\nabla^4\gamma)(t_0)$  are linearly independent, then  $\nabla\text{-Tan}(\gamma)$  is locally diffeomorphic to the folded umbrella at  $(t_0, 0) \in I \times \mathbf{R}$ .*

(2) *Let  $\dim(M) = 3$ . If  $(\nabla\gamma)(t_0) = 0$  and  $(\nabla^2\gamma)(t_0), (\nabla^3\gamma)(t_0), (\nabla^4\gamma)(t_0)$  are linearly independent, then  $\nabla\text{-Tan}(\gamma)$  is locally diffeomorphic to the swallowtail at  $(t_0, 0) \in I \times \mathbf{R}$ .*

(3) *Let  $\dim(M) \geq 4$ . If  $(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), (\nabla^3\gamma)(t_0)$  are linearly independent, then the  $\nabla$ -tangent surface  $\nabla\text{-Tan}(\gamma)$  is locally diffeomorphic to the embedded cuspidal edge at  $(t_0, 0) \in I \times \mathbf{R}$ .*

(4) *Let  $\dim(M) \geq 4$ . If  $(\nabla\gamma)(t_0) = 0$  and  $(\nabla^2\gamma)(t_0), (\nabla^3\gamma)(t_0), (\nabla^4\gamma)(t_0), (\nabla^5\gamma)(t_0)$  are linearly independent, then  $\nabla\text{-Tan}(\gamma)$  is locally diffeomorphic to the open swallowtail at  $(t_0, 0) \in I \times \mathbf{R}$ .*

A map-germ  $f : (\mathbf{R}^2, p) \rightarrow M$  is locally diffeomorphic at  $p$  to another map-germ  $g : (\mathbf{R}^2, p') \rightarrow M'$  if there exist diffeomorphism-germs  $\sigma : (\mathbf{R}^2, p) \rightarrow (\mathbf{R}^2, p')$  and  $\tau : (M, f(p)) \rightarrow (M', g(p'))$  such that  $\tau \circ f = g \circ \sigma : (\mathbf{R}^2, p) \rightarrow (M', g(p'))$ .

The *cuspidal edge* is defined by the map-germ  $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^m, 0)$ ,  $m \geq 3$ ,

$$(t, s) \mapsto (t + s, t^2 + 2st, t^3 + 3st^2, 0, \dots, 0),$$

which is diffeomorphic to  $(u, w) \mapsto (u, w^2, w^3, 0, \dots, 0)$ . The cuspidal edge singularities are originally defined only in the three dimensional space. Here we are generalizing the notion of the cuspidal edge in higher dimensional space. In Theorem 1.3 (2), we emphasize it by writing “embedded” cuspidal edge. In what follows, we call it just cuspidal edge for simplicity even in the case  $m \geq 4$ . The *folded umbrella* (or the *cuspidal cross cap*) is defined by the map-germ  $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ ,

$$(t, s) \mapsto (t + s, t^2 + 2st, t^4 + 4st^3),$$

which is diffeomorphic to  $(u, t) \mapsto (u, t^2 + ut, t^4 + \frac{2}{3}ut)$ . The *swallowtail* is defined by the map-germ  $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$

$$(t, s) \mapsto (t^2 + s, t^3 + \frac{3}{2}st, t^4 + 2st^2),$$

which is diffeomorphic to  $(u, t) \mapsto (u, t^3 + ut, t^4 + \frac{2}{3}ut^2)$ . The *open swallowtail* is defined by the map-germ  $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^m, 0)$ ,  $m \geq 4$ ,

$$(t, s) \mapsto (t^2 + s, t^3 + \frac{3}{2}st, t^4 + 2st^2, t^5 + \frac{5}{2}st^3, 0, \dots, 0),$$

which is diffeomorphic to  $(u, t) \mapsto (u, t^3 + ut, t^4 + \frac{2}{3}ut^2, t^5 + \frac{5}{9}ut^3, 0, \dots, 0)$ . The open swallowtail singularity was introduced by Arnol'd (see [2]) as a singularity of Lagrangian varieties in symplectic geometry. Here we abstract its diffeomorphism class as the singularity of tangent surfaces (see [13][19]).

In [23], Izumiya, Nagai, Saji introduced and studied the class “E-flat” great circular surfaces in the standard three sphere  $S^3$  in detail, which contains the class of tangent surfaces to curves in  $S^3$ . The generic classifications given there (Theorems 1.2, 1.3 of [23]) in the sphere geometry

become different from ours, because of the differences of topology and mappings spaces defining the genericity.

Because we treat singularities in a general ambient space, we need the intrinsic characterizations of singularities found in [25][12]. Note that the characterization of swallowtails was applied to hyperbolic geometry in [25] and to Euclidean and affine geometries in [20]. The characterization of folded umbrellas is applied to Lorenz-Minkowski geometry in [12]. In this paper we apply to non-flat projective geometry the characterizations and their some generalization via the notion of openings introduced by the first author ([19], see also [17]).

In §2, we recall on affine connections and related notions. In §3, we define the tangent surface to an immersed curve and show that the tangent surface is a frontal under certain general conditions (Lemma 3.1). In §4 we introduce the notion of directed curves and define their tangent surfaces. We recall the criteria of singularities in §5 and §6. After a preliminary calculations in §7, we show Theorem 1.3 in §8, using the criteria of singularities for the Euclidean case and in §9 in general. In §10 we show Theorem 1.1, and in §11 we prove Theorem 1.2. Apart from main theorems, but related to them, we give an observation on the singularities of tangent surfaces to torsionless curves in §12. In flat case the tangent surface to a torsionless curve necessarily has the fold singularity. However, in non-flat case we have an example where (2,5)-cuspidal edge singularity appears on the tangent surface of some torsionless curve.

In this paper all manifolds and mappings are assumed to be of class  $C^\infty$  unless otherwise stated.

## 2 Affine connection and geodesic

Let  $M$  be an  $m$ -dimensional manifold with an affine connection  $\nabla$  (see [15][24]). For any vector fields  $X, Y$  on  $M$ , a vector field  $\nabla_X Y$  on  $M$ , which is called the *covariant derivative* of  $Y$  by  $X$ , is assigned such that

$$\nabla_{hX+kY}Z = h\nabla_X Z + k\nabla_Y Z, \quad \nabla_X(hY + kZ) = h\nabla_X Y + (Xh)Y + k\nabla_X Z + (Xk)Z,$$

for any vector fields  $X, Y, Z$  and functions  $h, k$  on  $M$ .

For a system of local coordinates  $x^\lambda$  ( $\lambda = 1, 2, \dots, m$ ), we write

$$\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = \Gamma_{\mu\nu}^\lambda \frac{\partial}{\partial x^\lambda},$$

using the Christoffel symbols (coefficients of the connection)  $\Gamma_{\mu\nu}^\lambda$  and the Einstein convention.

In general, for a given mapping  $g : N \rightarrow (M, \nabla)$ , we define the notion of *covariant derivative*  $\nabla_\eta^g v : N \rightarrow TM$  of a vector field  $v : N \rightarrow TM$  along  $g$  by a vector field  $\eta : N \rightarrow TN$  over a manifold  $N$  (see [12]). Using a local presentation  $\eta(t) = \eta^i(t) \frac{\partial}{\partial t^i}(t)$ ,  $v(t) = v^\lambda(t) \frac{\partial}{\partial x^\lambda}(g(t))$ , for local coordinates  $t = (t_1, \dots, t_n)$  of  $N$  and  $x = (x_1, \dots, x_m)$  of  $M$ , and using the Einstein convention, we define

$$(\nabla_\eta^g v)(t) := \left\{ \eta^i(t) \frac{\partial v^\lambda}{\partial t^i}(t) + \Gamma_{\mu\nu}^\lambda(g(t)) \eta^i(t) \frac{\partial g^\mu}{\partial t^i}(t) v^\nu(t) \right\} \frac{\partial}{\partial x^\lambda}(g(t)).$$

Here  $g^\mu = x^\mu \circ g$ . The definition is naturally derived from the parallelisms on  $M$  induced by the connection  $\nabla$ .

Then we have, as in the usual case,

$$\nabla_{h\eta+k\xi}^f v = h\nabla_\eta^f v + k\nabla_\xi^f v, \quad \nabla_\eta^f(hv + kw) = h\nabla_\eta^f v + k\nabla_\eta^f w + (\eta h)v + (\eta k)w,$$

for any vector fields  $v, w$  along  $f$ ,  $\eta, \xi$  over  $N$ , and functions  $h, k$  on  $N$ .

If  $g : M \rightarrow M$  is the identity mapping, then  $\nabla_\eta^g v$  is just the ordinary covariant derivative  $\nabla_\eta v$  for vector fields  $\eta, v$  over  $M$ . The covariant derivative along a mapping is well-defined also for any tensor field over the mapping, that is compatible with any contractions.

Then we get the notion of geodesics: A curve  $\varphi : I \rightarrow M, \varphi = \varphi(s)$  is called a  $\nabla$ -geodesic if  $\nabla_{\partial/\partial s}^\varphi \frac{d\varphi}{ds} = 0$ . For  $x \in M, v \in T_x M$ , let  $\varphi(x, v, s)$  denote the  $\nabla$ -geodesic determined by the differential equation

$$\frac{\partial^2 \varphi^\lambda}{\partial s^2}(x, v, s) + \Gamma_{\mu\nu}^\lambda(\varphi(x, v, s)) \frac{\partial \varphi^\mu}{\partial s}(x, v, s) \frac{\partial \varphi^\nu}{\partial s}(x, v, s) = 0,$$

with the initial conditions  $\varphi(x, v, 0) = x$  and  $\frac{\partial \varphi}{\partial s}(x, v, 0) = v$ . Here  $\varphi^\lambda$  denotes the  $\lambda$ -th component  $x^\lambda \circ \varphi$  of  $\varphi$ . Note that  $\varphi : U(\subset TM \times \mathbf{R}) \rightarrow M$  is defined on an open neighborhood  $U$  of  $TM \times \{0\}$ .

**Lemma 2.1** *At each point of  $M$ , there exists an open neighborhood  $U$  of the point such that the  $\nabla$ -geodesics  $\varphi(x, v, s)$  are written*

$$\varphi(x, v, s) = x + sv + \frac{1}{2}s^2 h(x, v, s)$$

for some  $C^\infty$  mapping  $h(x, v, s)$  on an open neighborhood of  $TU \times \{0\}$  in  $TU \times \mathbf{R}$ .

*Proof:* Since  $\varphi(x, v, 0) = x$ , we have locally  $\varphi(x, v, s) = x + s\tilde{\varphi}(x, v, s)$  for some  $C^\infty$  mapping  $\tilde{\varphi}$ . Then  $\frac{\partial \varphi}{\partial s}(x, v, s) = \tilde{\varphi}(x, v, s) + s\frac{\partial \tilde{\varphi}}{\partial s}(x, v, s)$ . Since  $\tilde{\varphi}(x, v, 0) = \frac{\partial \varphi}{\partial s}(x, v, 0) = v$ , we have locally  $\tilde{\varphi}(x, v, s) = v + \frac{1}{2}s h(x, v, s)$  for some  $C^\infty$  mapping  $h$ . Thus we have the result.  $\square$

**Lemma 2.2** *Let  $\varphi(x, v, s) = x + sv + \frac{1}{2}s^2 h(x, v, s)$  be the expression of the  $\nabla$ -geodesics as in Lemma 2.1. Then we have*

$$\begin{aligned} h^\lambda(x, v, 0) &= \frac{\partial^2 \varphi^\lambda}{\partial s^2}(x, v, 0) = -\Gamma_{\mu\nu}^\lambda(x)v^\mu v^\nu, \\ \frac{\partial h^\lambda}{\partial x^\kappa}(x, v, 0) &= -\Gamma_{\mu\nu, \kappa}^\lambda(x)v^\mu v^\nu, \\ \frac{\partial h^\lambda}{\partial v^\rho}(x, v, 0) &= -\Gamma_{\rho\nu}^\lambda(x)v^\nu - \Gamma_{\mu\rho}^\lambda(x)v^\mu, \\ \frac{\partial h^\lambda}{\partial s}(x, v, 0) &= \frac{1}{3}\frac{\partial^3 \varphi}{\partial s^3}(x, v, 0) = \frac{1}{3}(-\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\rho\kappa}^\lambda \Gamma_{\mu\nu}^\rho + \Gamma_{\kappa\rho}^\lambda \Gamma_{\mu\nu}^\rho)v^\mu v^\nu v^\kappa \end{aligned}$$

where  $\Gamma_{\mu\nu, \kappa}^\lambda = \partial \Gamma_{\mu\nu}^\lambda / \partial x^\kappa$ .

*Proof:* For a fixed  $(x, v)$ ,  $\varphi(x, v, s)$  is a  $\nabla$ -geodesic, therefore we have

$$\frac{\partial^2 \varphi^\lambda}{\partial s^2}(x, v, s) = -\Gamma_{\mu\nu}^\lambda(\varphi(x, v, s)) \frac{\partial \varphi^\mu}{\partial s}(x, v, s) \frac{\partial \varphi^\nu}{\partial s}(x, v, s).$$

Since  $\frac{\partial \varphi}{\partial s}(x, v, 0) = v$ , we have that  $\frac{\partial^2 \varphi}{\partial s^2}(x, v, 0) = h(x, v, 0)$ , and that  $h^\lambda(x, v, 0) = -\Gamma_{\mu\nu}^\lambda(x)v^\mu v^\nu$ . Moreover we have

$$\begin{aligned} \frac{\partial^3 \varphi^\lambda}{\partial s^3}(x, v, s) &= -\Gamma_{\mu\nu, \kappa}^\lambda(\varphi(x, v, s)) \frac{\partial \varphi^\kappa}{\partial s}(x, v, s) \frac{\partial \varphi^\mu}{\partial s}(x, v, s) \frac{\partial \varphi^\nu}{\partial s}(x, v, s) \\ &\quad - \Gamma_{\mu\nu}^\lambda \frac{\partial^2 \varphi^\mu}{\partial s^2}(x, v, s) \frac{\partial \varphi^\nu}{\partial s}(x, v, s) - \Gamma_{\mu\nu}^\lambda \frac{\partial \varphi^\mu}{\partial s}(x, v, s) \frac{\partial^2 \varphi^\nu}{\partial s^2}(x, v, s). \end{aligned}$$

By setting  $s = 0$  and by the first equality, we have the fourth equality. By differentiating both sides of  $h^\lambda(x, v, 0) = -\Gamma_{\mu\nu}^\lambda(x)v^\mu v^\nu$  (the first equality), by  $x^\kappa$  and  $v^\rho$  we have the second and the third equalities.  $\square$

A connection  $\nabla$  on  $M$  is called *torsion free* if  $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0$ , for any vector fields  $X, Y$  on  $M$ . A connection  $\nabla$  is torsion free if and only if, for any system of local coordinates,  $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda, 1 \leq \lambda, \mu, \nu \leq m$ .

**Remark 2.3** We observe that the equation on geodesics

$$\frac{\partial^2 \varphi^\lambda}{\partial s^2}(x, v, s) + \Gamma_{\mu\nu}^\lambda(\varphi(x, v, s)) \frac{\partial \varphi^\mu}{\partial s}(x, v, s) \frac{\partial \varphi^\nu}{\partial s}(x, v, s) = 0,$$

is symmetric on the indices  $\mu, \nu$ . Therefore The geodesics  $\varphi(x, v, s)$  and the tangent surfaces  $\text{Tan}(\gamma)$  remain same if the connection  $\Gamma_{\mu\nu}^\lambda$  is replaced by the torsion free connection  $\frac{1}{2}(\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda)$ , in other word, if  $\nabla$  is replaced by the torsion free connection  $\tilde{\nabla}$ , defined by  $\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}T(X, Y)$ .

Let  $\gamma : I \rightarrow M$  be a curve which is not necessarily a geodesic nor an immersed curve. Then the first derivative  $(\nabla \gamma)(t)$  means just the velocity vector field  $\gamma'(t)$ . The second derivative  $(\nabla^2 \gamma)(t)$  is defined, in terms of covariant derivative along the curve  $\gamma$ , by

$$(\nabla^2 \gamma)(t) := \nabla_{\partial/\partial t}^\gamma (\nabla \gamma)(t).$$

Note that  $\gamma$  is a  $\nabla$ -geodesic if and only if  $\nabla^2 \gamma = 0$ . In general, we define  $k$ -th covariant derivative of  $\gamma$  inductively by

$$(\nabla^k \gamma)(t) := \nabla_{\partial/\partial t}^\gamma (\nabla^{k-1} \gamma)(t), \quad (k \geq 2).$$

Then we have by direct calculations:

**Lemma 2.4**

$$\begin{aligned} (\nabla \gamma)^\lambda &= (\gamma')^\lambda, \\ (\nabla^2 \gamma)^\lambda &= (\gamma'')^\lambda + \Gamma_{\mu\nu}^\lambda (\gamma')^\mu (\gamma')^\nu, \\ (\nabla^3 \gamma)^\lambda &= (\gamma''')^\lambda + (\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\kappa\rho}^\lambda \Gamma_{\mu\nu}^\rho) (\gamma')^\mu (\gamma')^\nu (\gamma')^\kappa + (2\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda) (\gamma')^\mu (\gamma'')^\nu. \end{aligned}$$

### 3 Tangent surface and frontal

Let  $\gamma : I \rightarrow M$  be a  $C^\infty$  immersion from an open interval  $I$ . Then the tangent surface to  $\gamma$  is the ruled surface by tangent  $\nabla$ -geodesics to  $\gamma$ . More precisely the  $\nabla$ -tangent surface  $f = \nabla\text{-Tan}(\gamma) : V(\subset I \times \mathbf{R}) \rightarrow M$  to the curve  $\gamma$  is defined by

$$f(t, s) := \varphi(\gamma(t), \gamma'(t), s),$$

on an open neighborhood  $V$  of  $I \times \{0\}$ .

The mapping  $f$  has singularity at least along  $\{s = 0\}$ . In fact  $f_*(\frac{\partial}{\partial t})(t, 0) = \gamma'(t) = f_*(\frac{\partial}{\partial s})(t, 0)$  and the kernel of the differential  $f_*$  is generated by the vector field  $\eta = \frac{\partial}{\partial t} - \frac{\partial}{\partial s}$  along  $\{s = 0\}$  on the  $t$ - $s$ -plane.

A map-germ  $f : (\mathbf{R}^n, p) \rightarrow M$ ,  $n \leq m = \dim(M)$  is called a *frontal* if there exists a  $C^\infty$  integral lifting  $\tilde{f} : (\mathbf{R}^n, p) \rightarrow \text{Gr}(n, TM)$  of  $f$ . Here  $\text{Gr}(n, TM)$  means the Grassmannian bundle over  $M$  consisting of tangential  $n$ -planes in  $TM$ ,

$$\text{Gr}(n, TM) := \{\Pi \mid \Pi \subseteq T_x M, \Pi \text{ is a linear subspace, } \dim(\Pi) = n, x \in M\},$$

with the canonical projection  $\pi : \text{Gr}(n, TM) \rightarrow M$  to the base manifold  $M$ , and we call  $\tilde{f}$  *integral* if  $f_*(T_q \mathbf{R}^n) \subseteq \tilde{f}(q)$  for any  $q$  in a neighborhood of  $p$  in  $\mathbf{R}^n$ , after taking a representative of  $f$ . The definition generalizes the preceding definition of “frontal” in the case  $m = n + 1$  (see [12]). The definitions in the case  $m = n + 1$  are equivalent to each other as is easily seen. Note that, in [19], we have introduced the same notion of frontal mapping under the restriction that the locus of immersive points of  $f$  is dense, where the integral lifting  $\tilde{f}$  is uniquely determined.

Let  $f : (\mathbf{R}^n, p) \rightarrow M$  is a frontal and  $\tilde{f}$  is an integral lifting of  $f$ . Then there exists a frame  $V_1, V_2, \dots, V_n : (\mathbf{R}^n, p) \rightarrow TM$  along  $f$  associated with  $\tilde{f}$  such that

$$\tilde{f}(q) = \langle V_1(q), V_2(q), \dots, V_n(q) \rangle_{\mathbf{R}},$$

for any  $q$  in a neighborhood of  $p$  in  $\mathbf{R}^n$ . Then there is a  $C^\infty$  function-germ  $\sigma : (\mathbf{R}^n, p) \rightarrow \mathbf{R}$  such that

$$\left( \frac{\partial f}{\partial t_1} \wedge \frac{\partial f}{\partial t_2} \wedge \dots \wedge \frac{\partial f}{\partial t_n} \right)(t) = \sigma(t)(V_1 \wedge V_2 \wedge \dots \wedge V_n)(t),$$

as germs of  $n$ -vector fields  $(\mathbf{R}^n, p) \rightarrow \wedge^n TM$  over  $f$ . Then the singular locus (non-immersive locus)  $S(f)$  of  $f$  coincides with the zero locus  $\{\sigma = 0\}$  of  $\sigma$ . We call  $\sigma$  a *signed area density function* or briefly an *s-function* of the frontal  $f$ . Note that the function  $\sigma$  is essentially the same thing with the function  $\lambda$  introduced in [25][12] in the case  $\dim(M) = 3$ . However we avoid the notation  $\lambda$  here because we use it for index of Christoffel symbols.

We say that a frontal  $f : (\mathbf{R}^n, p) \rightarrow M$  has a *non-degenerate* singular point at  $p$  if the  $s$ -function  $\sigma$  of  $f$  satisfies  $\sigma(p) = 0$  and  $d\sigma(p) \neq 0$ . The condition is independent of the choice of the integral lifting  $\tilde{f}$  and the associated frame  $V_1, V_2, \dots, V_n$ . If  $f$  has a non-degenerate singular point at  $p$ , then  $f$  is of corank 1 such that the singular locus  $S(f) \subset (\mathbf{R}^n, p)$  is a regular hypersurface.

In this paper we concern with only the cases  $n = 1$  and  $n = 2$ .

Returning to our situation, we have

**Lemma 3.1** *Suppose  $(\nabla\gamma)(t_0)$  and  $(\nabla^2\gamma)(t_0)$  are linearly independent. Then the germ of tangent surface  $\nabla\text{-Tan}(\gamma)$  is a frontal with the non-degenerate singular point at  $(t_0, 0)$  and with the singular locus  $S(\nabla\text{-Tan}(\gamma)) = \{s = 0\}$ .*

*Proof:* We set  $f(t, s) = \text{Tan}(\gamma)(t, s) = \varphi(\gamma(t), \gamma'(t), s)$ . By Lemma 2.1, write  $\varphi = x + s v + \frac{1}{2}s^2 h$ . Then we have

$$f(t, s) = \gamma(t) + s \gamma'(t) + \frac{1}{2}s^2 h(\gamma(t), \gamma'(t), s),$$

and

$$\begin{aligned} \frac{\partial f}{\partial t} &= \gamma' + s \gamma'' + \frac{1}{2}s^2 (\gamma')^\mu \frac{\partial h}{\partial x^\mu}(\gamma, \gamma', s) + \frac{1}{2}s^2 (\gamma'')^\nu \frac{\partial h}{\partial v^\nu}(\gamma, \gamma', s), \\ \frac{\partial f}{\partial s} &= \gamma' + s h(\gamma, \gamma', s) + \frac{1}{2}s^2 \frac{\partial h}{\partial s}(\gamma, \gamma', s). \end{aligned}$$

Then we see that  $S(f) \supseteq \{s = 0\}$ .

Let  $s \neq 0$ . Then

$$\begin{aligned} \frac{1}{s} \left( \frac{\partial f}{\partial t} - \frac{\partial f}{\partial s} \right) &= \gamma'' + \frac{1}{2}s (\gamma')^\mu \frac{\partial h}{\partial x^\mu}(\gamma, \gamma', s) + \frac{1}{2}s (\gamma'')^\nu \frac{\partial h}{\partial v^\nu}(\gamma, \gamma', s) \\ &\quad - h(\gamma, \gamma', s) - \frac{1}{2}s \frac{\partial h}{\partial s}(\gamma, \gamma', s). \end{aligned}$$

We define  $F(t, s)$  by the right hand side. Then  $F(t, s) = \frac{1}{s} \left( \frac{\partial f}{\partial t} - \frac{\partial f}{\partial s} \right)$  if  $s \neq 0$ . Moreover  $F$  is  $C^\infty$  also on  $s = 0$  and

$$F(t, 0) = \gamma''(t) - h(\gamma(t), \gamma'(t), 0).$$

By Lemma 2.2, we have

$$h^\lambda(\gamma(t), \gamma'(t), 0) = -\Gamma_{\mu\nu}^\lambda(\gamma(t)) (\gamma'(t))^\mu (\gamma'(t))^\nu.$$

Hence we have

$$F^\lambda(t, 0) = (\gamma''(t))^\lambda + \Gamma_{\mu\nu}^\lambda(\gamma(t)) (\gamma'(t))^\mu (\gamma'(t))^\nu = (\nabla^2 \gamma)^\lambda(t).$$

Therefore if  $(\nabla \gamma)(t), (\nabla^2 \gamma)(t)$  are linearly independent at  $t = t_0$ , then  $\frac{\partial f}{\partial t}(t, s)$  and  $F(t, s)$  are linearly independent around  $(t_0, 0)$  and satisfies that

$$\left( \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} \right)(t, s) = -s \left( \frac{\partial f}{\partial t} \wedge F \right)(t, s).$$

Therefore we see that  $\frac{\partial f}{\partial t}(t, s)$  and  $F(t, s)$  define the integral lifting of  $f$ ,  $f$  is frontal with non-degenerate singular point at  $(t_0, 0)$ , and that  $S(f) = \{s = 0\}$ .  $\square$

**Remark 3.2** In the Euclidean case, Lemma 3.1 holds globally on  $I \times \mathbf{R}$ . However, even in a locally projectively flat case, Lemma 3.1 holds just locally near  $I \times \{0\}$ . For example, let  $M$  be the standard three dimensional sphere  $S^3 \subset \mathbf{R}^4$  with the standard (Levi-Civita) connection. Then geodesics in  $S^3$  are given by great circles (with periodic parametrizations) and we observe, via the natural double covering  $S^3 \rightarrow \mathbf{R}P^3$ , that the tangent surface to any curve in  $S^3$  has singularities not only along the original curve, but also along the antipodal of the curve (cf. [23]).

## 4 Tangent surface to directed curve

Let  $PTM = \text{Gr}(1, TM)$  denote the projective tangent bundle over the manifold  $M$ , and  $\pi : PTM \rightarrow M$  the natural projection. The fiber of  $\pi$  over  $x \in M$  is the projective space  $P(T_x M)$  of dimension  $m - 1$ .

A curve  $\gamma : I \rightarrow M$ , which is not necessarily an immersion, is called *directed* if there assigned a  $C^\infty$  lifting  $\tilde{\gamma} : I \rightarrow PTM$  of  $\gamma$  for  $\pi$  which satisfies  $\gamma'(t) \in \tilde{\gamma}(t) \subset T_{\gamma(t)}M$  for any  $t \in I$ . Here  $\tilde{\gamma}(t) \in P(T_{\gamma(t)}M)$  is regarded as a one-dimensional linear subspace of  $T_{\gamma(t)}M$ . The notion of directed curves is nothing but the notion of frontal maps introduced in §3 in the case  $n = 1$  with assignment of an integral lifting. Then we regard the direction  $\tilde{\gamma}(t_0)$  is assigned to each point  $\gamma(t_0)$  on  $\gamma$ . Note that if  $\gamma'(t_0) \neq 0$ , then  $\tilde{\gamma}(t_0)$  is uniquely determined by the tangent line  $\langle \gamma'(t_0) \rangle_{\mathbf{R}} \subset T_{\gamma(t_0)}M$ .

Let  $\gamma : I \rightarrow M$  be a directed curve and  $\tilde{\gamma}$  its integral lifting. Then there exists a frame  $u : I \rightarrow TM$  of  $\tilde{\gamma}$  which satisfies  $\tilde{\gamma}(t) = \langle u(t) \rangle_{\mathbf{R}}, u(t) \neq 0$  for any  $t \in I$ . Note that there exists a unique function  $a(t)$  such that  $\gamma'(t) = a(t)u(t)$ . Then define the  $\nabla$ -tangent surface  $f = \nabla\text{-Tan}(\gamma) : V(\subset I \times \mathbf{R}) \rightarrow M$  by

$$f(t, s) := \varphi(\gamma(t), u(t), s),$$

using the family of  $\nabla$ -geodesics  $\varphi = \varphi(x, v, s)$  and a frame  $u(t)$ . The  $\nabla$ -tangent surface for an immersed curve  $\gamma$  in §3 was defined by the frame  $u(t) = \gamma'(t)$ .

**Lemma 4.1** *If the immersion locus of a directed curve  $\gamma : I \rightarrow M$  is dense in  $I$ , then the integral lifting  $\tilde{\gamma}$  is uniquely determined. The right equivalence class of the germ of  $\nabla\text{-Tan}(\gamma) : (I \times \mathbf{R}, I \times \{0\}) \rightarrow M$  for a directed curve  $\gamma$  is independent of the choice of the frame  $u$ .*

*Proof:* The first half is clear because  $\tilde{\gamma}$  is  $C^\infty$ , so is continuous. The second half is achieved by the diffeomorphism  $(t, s) \rightarrow (t, c(t)s)$  for another choice  $c(t)u(t), c(t) \neq 0$ .  $\square$

**Lemma 4.2** *Let  $k \geq 2$ . Suppose  $(\nabla\gamma)^i(t_0) = 0, 1 \leq i < k$  and  $(\nabla\gamma)^k(t_0) \neq 0$ . Then we have:*

(1) *For any coordinates of  $M$  around  $\gamma(t_0)$ ,  $\gamma^{(i)}(t_0) = 0, 1 \leq i < k$  and  $\gamma^{(k)}(t_0) = (\nabla^k\gamma)(t_0) \neq 0$ . Moreover we have  $\gamma^{(k+1)}(t_0) = (\nabla^{k+1}\gamma)(t_0)$ .*

(2) *Set*

$$u(t) = \frac{1}{k(t - t_0)^{k-1}} \gamma'(t).$$

*Then  $u$  is a  $C^\infty$  vector field along  $\gamma$  on a neighborhood of  $t_0$ . The curve  $\gamma$  is directed on a neighborhood of  $t_0$  by the frame  $u$ .*

(3) *For any frame  $u(t)$  of the directed curve  $\gamma$  around  $t_0$ , and for any  $\ell \geq 0$ ,*

$$(\nabla^k\gamma)(t_0), (\nabla^{k+1}\gamma)(t_0), \dots, (\nabla^{k+\ell}\gamma)(t_0)$$

*are linearly independent if and only if*

$$u(t_0), (\nabla_{\partial/\partial t}^\gamma u)(t_0), \dots, ((\nabla_{\partial/\partial t}^\gamma)^\ell u)(t_0)$$

*are linearly independent. In particular, for the frame in (2), we have*

$$u(t_0) = \frac{1}{k!}(\nabla^k\gamma)(t_0), (\nabla u)(t_0) = \frac{1}{k \cdot k!}(\nabla^{k+1}\gamma)(t_0), \dots, (\nabla^\ell u)(t_0) = \frac{\ell!}{k \cdot (k + \ell - 1)!}(\nabla^{k+\ell}\gamma)(t_0).$$

*where  $\nabla^i u = (\nabla_{\partial/\partial t}^\gamma)^i u$ .*

*Proof:* (1) Let  $k = 2$ . Then  $\gamma'(t_0) = (\nabla\gamma)(t_0) = 0$ . By Lemma 2.4, we have  $\gamma''(t_0) = (\nabla^2\gamma)(t_0) \neq 0$ ,  $\gamma'''(t_0) = (\nabla^3\gamma)(t_0)$ . Let  $k \geq 3$ . Then  $(\nabla^k\gamma)^\lambda$  is a sum of  $(\gamma^{(k)})^\lambda$  and a polynomial of  $\Gamma_{\mu\nu}^\lambda$ , their partial derivatives and  $\gamma^{(i)}$ ,  $i < k$ , each monomial of which contains a  $\gamma^{(i)}$  with  $i \leq k-2$  (cf. Lemma 2.4). Thus we have  $\gamma^{(i)}(t_0) = (\nabla\gamma)^i(t_0) = 0$ ,  $1 \leq i < k$ . Moreover we have  $0 \neq (\nabla^k\gamma)(t_0) = \gamma^{(k)}(t_0)$  and  $(\nabla^{k+1}\gamma)(t_0) = \gamma^{(k+1)}(t_0)$ .

(2) is clear.

(3) We have that  $c(t)u(t) = \gamma'(t)$  for some function  $c(t)$ . If  $k \geq 2$ , then  $c(t_0) = 0$ . By operating  $\nabla_{\partial/\partial t}^\gamma$  to both sides of  $c(t)u(t) = \gamma'(t)$ , we have

$$c'(t)u(t) + c(t)(\nabla_{\partial/\partial t}^\gamma u)(t) = (\nabla^2\gamma)(t).$$

If  $k \geq 3$ , then  $c(t_0) = 0, c'(t_0) = 0$ . In general we have

$$c(t_0) = c'(t_0) = \dots = c^{(k-2)}(t_0) = 0, c^{(k-1)}(t_0) \neq 0,$$

and

$$\begin{aligned} c^{(k-1)}(t)u(t) + (k-1)c^{(k-2)}(t)(\nabla u)(t) + {}_{k-1}C_2c^{(k-3)}(t)(\nabla^2u)(t) + \dots &= (\nabla^k\gamma)(t) \\ c^{(k)}(t)u(t) + kc^{(k-1)}(t)(\nabla u)(t) + {}_kC_2c^{(k-2)}(t)(\nabla^2u)(t) + \dots &= (\nabla^{k+1}\gamma)(t) \\ \vdots &\vdots \\ c^{(k+\ell)}(t)u(t) + \dots + {}_{k+\ell-1}C_{k-1}c^{(k-1)}(t)(\nabla^\ell u)(t) + \dots &= (\nabla^{k+\ell}\gamma)(t). \end{aligned}$$

Evaluating at  $t_0$ , we have the result.  $\square$

Lemma 3.1 is generalized as follows:

**Lemma 4.3** *Let  $\gamma : I \rightarrow M$  be a  $C^\infty$  curve,  $t_0 \in I$ , and  $k \geq 1$ . Suppose that  $(\nabla^i\gamma)(t_0) = 0$ ,  $1 \leq i < k$  and that  $(\nabla^k\gamma)(t_0), (\nabla^{k+1}\gamma)(t_0)$  are linearly independent. Then the germ of  $\nabla\text{-Tan}(\gamma)$  is a frontal with non-degenerate singular point at  $(t_0, 0)$  and with the singular locus  $S(\nabla\text{-Tan}(\gamma)) = \{s = 0\}$ .*

*Proof:* Suppose  $k \geq 2$ . Let  $u(t)$  be a frame around  $t_0$  of the directed curve  $\gamma$  and  $c(t)u(t) = \gamma'(t)$ ,  $u(t_0) \neq 0$ . (For instance  $c(t) = k(t - t_0)^{k-1}$ ). Since  $f(t, s) = \gamma(t) + su(t) + \frac{1}{2}s^2h(\gamma(t), u(t), s)$ , we have

$$\begin{aligned} \frac{\partial f}{\partial t} &= \gamma' + su' + \frac{1}{2}s^2(\gamma')^\mu \frac{\partial h}{\partial x^\mu}(\gamma, u, s) + \frac{1}{2}s^2(u')^\nu \frac{\partial h}{\partial v^\nu}(\gamma, u, s), \\ \frac{\partial f}{\partial s} &= u + sh(\gamma, u, s) + \frac{1}{2}s^2 \frac{\partial h}{\partial s}(\gamma, u, s). \end{aligned}$$

Then we see that  $S(f) \supseteq \{s = 0\}$  and the kernel field of  $f_*$  along  $\{s = 0\}$  is given by  $\eta = \frac{\partial}{\partial t} - c(t)\frac{\partial}{\partial s}$ . Let  $s \neq 0$ . Then

$$\begin{aligned} \frac{1}{s}(\frac{\partial f}{\partial t} - c(t)\frac{\partial f}{\partial s}) &= u' + \frac{1}{2}s(\gamma')^\mu \frac{\partial h}{\partial x^\mu}(\gamma, u, s) + \frac{1}{2}s(u')^\nu \frac{\partial h}{\partial v^\nu}(\gamma, u, s) \\ &\quad - c(t)h(\gamma, u, s) - \frac{1}{2}sc(t)\frac{\partial h}{\partial s}(\gamma, u, s). \end{aligned}$$

We define  $F(t, s)$  by the right hand side. Then  $F(t, s) = \frac{1}{s}(\frac{\partial f}{\partial t} - c(t)\frac{\partial f}{\partial s})$  if  $s \neq 0$ . Moreover  $F$  is  $C^\infty$  also on  $s = 0$  and

$$F(t, 0) = u'(t) - c(t)h(\gamma(t), u(t), 0).$$

By Lemma 2.2,

$$F(t, 0) = u'(t) + c(t)\Gamma_{\mu\nu}^\lambda(\gamma(t))(u(t))^\mu(u(t))^\nu = (\nabla_{\partial/\partial t}^\gamma u)(t).$$

By Lemma 4.2 (3), if  $(\nabla^k \gamma)(t_0), (\nabla^{k+1} \gamma)(t_0)$  are linearly independent, then  $\frac{\partial f}{\partial s}(t, s)$  and  $F(t, s)$  are linearly independent around  $(t_0, 0)$ . Moreover they satisfies

$$(\frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s})(t, s) = -s(\frac{\partial f}{\partial s} \wedge F)(t, s).$$

Therefore we see that  $\frac{\partial f}{\partial s}(t, s)$  and  $F(t, s)$  define an integral lifting of  $f$ ,  $f$  is frontal with non-degenerate singular point at  $(t_0, 0)$ , and that  $S(f) = \{s = 0\}$ .  $\square$

## 5 Cuspidal edge and folded umbrella

Let  $f : (\mathbf{R}^2, p) \rightarrow M^3$  be a frontal with a non-degenerate singular point at  $p$  and  $\tilde{f} : (\mathbf{R}^2, p) \rightarrow \text{Gr}(2, TM)$  the integral lifting of  $f$ . Let  $V_1, V_2 : (\mathbf{R}^2, p) \rightarrow TM$  be an associated frame with  $\tilde{f}$ . Let  $L : (\mathbf{R}^2, p) \rightarrow T^*M \setminus \zeta$  be an annihilator of  $\tilde{f}$ . The condition is that  $\langle L, V_1 \rangle = 0, \langle L, V_2 \rangle = 0$ . Here  $\zeta$  means the zero section. Let  $c : (\mathbf{R}, t_0) \rightarrow (\mathbf{R}^2, p)$  be a parametrization of the singular locus  $S(f)$ ,  $p = c(t_0)$ , and  $\eta : (\mathbf{R}^2, p) \rightarrow T\mathbf{R}^2$  be a vector field which restricts to the kernel field of  $f_*$  on  $S(f)$ . Suppose that  $V_2(p) \notin f_*(T_p \mathbf{R}^2)$ . Then, for any affine connection  $\nabla$  on  $M$ , we define

$$\psi(t) := \langle L(c(t)), (\nabla_\eta^f V_2)(c(t)) \rangle.$$

Note that the vector field  $(\nabla_\eta^f V_2)(c(t))$  is independent of the extension  $\eta$  and the choice of affine connection  $\nabla$ , since  $\eta|_{S(f)}$  is a kernel field of  $f_*$ . We call the function  $\psi(t)$  the *characteristic function* of  $f$ .

Then the following characterizations of cuspidal edges and folded umbrellas are given in [25][12]:

**Theorem 5.1** (Theorem 1.4 of [12]). *Let  $f : (\mathbf{R}^2, p) \rightarrow M^3$  be a germ of frontal with a non-degenerate singular point at  $p$ . Let  $c : (\mathbf{R}, t_0) \rightarrow (\mathbf{R}^2, p)$  be a parametrization of the singular locus of  $f$ . Suppose  $f_*c'(t_0) \neq 0$ . Then, for the characteristic function  $\psi$ ,*

- (1)  *$f$  is diffeomorphic to the cuspidal edge if and only if  $\psi(t_0) \neq 0$ .*
- (2)  *$f$  is diffeomorphic to the folded umbrella if and only if  $\psi(t_0) = 0, \psi'(t_0) \neq 0$ .*

Note that the conditions appeared in Theorem 5.1 are invariant under diffeomorphism equivalence.

**Remark 5.2** In the situation of Theorem 5.1, we set  $\gamma(t) = f(c(t))$ . Then we have  $\psi(t_0) \neq 0$  if and only if  $V_1(c(t_0)), V_2(c(t_0)), (\nabla_\eta^f V_2)(c(t_0))$  are linearly independent.

The above construction is generalized to the case  $m = \dim(M) \geq 4$ . In general, let  $f : (\mathbf{R}^2, p) \rightarrow M^m, m \geq 4$ , be a frontal with a non-degenerate singular point at  $p$  and  $\tilde{f} : (\mathbf{R}^2, p) \rightarrow \text{Gr}(2, TM)$  the integral lifting of  $f$ . Let  $V_1, V_2 : (\mathbf{R}^2, p) \rightarrow TM$  be an associated frame with  $\tilde{f}$ . We take a coframe  $L_1, \dots, L_{m-2} : (\mathbf{R}^2, p) \rightarrow T^*M$  satisfying that

$$\langle L_i(t, s), V_1(t, s) \rangle = 0, \langle L_i(t, s), V_2(t, s) \rangle = 0, \quad (1 \leq i \leq m-2),$$

and that  $L_1(t, s), \dots, L_{m-2}(t, s)$  are linearly independent for  $(t, s) \in (\mathbf{R}^2, p)$ . We define the *characteristic (vector valued) function*  $\psi : (\mathbf{R}, t_0) \rightarrow \mathbf{R}^{m-2}$  by  $\psi = (\psi_1, \dots, \psi_{m-2})$ ,

$$\psi_i(t) := \langle L_i(c(t)), (\nabla_\eta^f V_2)(c(t)) \rangle.$$

Let  $g : (\mathbf{R}^n, p) \rightarrow (\mathbf{R}^\ell, q)$  be a map-germ. A map germ  $f : (\mathbf{R}^n, p) \rightarrow \mathbf{R}^{\ell+r}$  is called an *opening* of  $g$  if there exist functions  $h_1, \dots, h_r : (\mathbf{R}^n, p) \rightarrow \mathbf{R}$  and functions  $a_{ij} : (\mathbf{R}^n, p) \rightarrow \mathbf{R}$ ,  $(1 \leq i \leq r, 1 \leq j \leq \ell)$  such that

$$dh_i = \sum_{j=1}^{\ell} a_{ij} dg_j, \quad f = (g, h_1, \dots, h_r),$$

(see for example [19]). If  $\ell = n$ , then the condition on  $h$  is equivalent to that  $f$  is frontal associated with an integral lifting  $\tilde{f} : (\mathbf{R}^n, p) \rightarrow \text{Gr}(n, T\mathbf{R}^{n+r})$  having Grassmannian coordinates  $(a_{ij})$  such that  $\tilde{f}(p)$  projects isomorphically to  $T_{g(p)}\mathbf{R}^n$  by the projection  $\mathbf{R}^{n+r} = \mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^n$ .

**Remark 5.3** A map-germ  $f : (\mathbf{R}^n, p) \rightarrow M$  is a frontal with a non-degenerate singular point at  $p$  if and only if  $f$  is diffeomorphic to an opening of a map-germ  $g : (\mathbf{R}^n, p) \rightarrow (\mathbf{R}^n, q)$  of Thom-Boardman singularity type  $\Sigma^1$  at  $p$ , i.e.  $g$  is of corank one and  $j^1 g : (\mathbf{R}^n, p) \rightarrow J^1(\mathbf{R}^n, \mathbf{R}^n)$  is transversal to the variety of singular 1-jets (see for example [14]).

We can summarize several known results as those on openings of the fold:

**Theorem 5.4** *Let  $f : (\mathbf{R}^2, p) \rightarrow M^m, m \geq 2$  be a germ of frontal with a non-degenerate singular point at  $p$ ,  $\tilde{f} : (\mathbf{R}^2, p) \rightarrow \text{Gr}(2, TM)$  the integral lifting of  $f$  and  $V_1, V_2 : (\mathbf{R}^2, p) \rightarrow TM$  an associated frame with  $\tilde{f}$ . Let  $c : (\mathbf{R}, t_0) \rightarrow (\mathbf{R}^2, p)$  be a parametrization of the singular locus of  $f$ . Suppose  $f_* c'(t_0) \neq 0$ . Then  $f$  is diffeomorphic to an opening of the fold, namely to the germ  $(u, w) \mapsto (u, \frac{1}{2}w^2)$ . Moreover we have:*

- (0) *Let  $m = 2$ . Then  $f$  is diffeomorphic to the fold.*
- (1) *Let  $m \geq 3$ . Then  $f$  is diffeomorphic to the cuspidal edge if and only if  $\psi(t_0) \neq 0$ .*
- (2) *Let  $m = 3$ . Then  $f$  is diffeomorphic to the folded umbrella if and only if  $\psi(t_0) = 0, \psi'(t_0) \neq 0$ .*

*Proof of Theorem 5.4:* The assertion (0) follows from Whitney's theorem (see [40][35][34]). (1) The condition  $\psi(t_0) \neq 0$  is equivalent to that  $f$  is a front, namely, that  $\tilde{f}$  is an immersion. Suppose  $\dim(M) = 3$ . Then by Proposition 1.3 of [25], we see that  $f$  is diffeomorphic to the cuspidal edge. In general cases  $m \geq 2$ , we see that there exists a submersion  $\pi : (M, f(p)) \rightarrow (\mathbf{R}^2, 0)$  such that

$$T_{f(p)}(\pi^{-1}(0)) + \tilde{f}(p) = T_{f(p)}M,$$

and that  $\pi \circ f$  satisfies the same condition with  $f$ , i.e.,  $\pi \circ f$  is a frontal with the non-degenerate singular point at  $p$  with the same singular locus with  $f$  and  $\eta(c(t_0))$  and  $c'(t_0)$  are linearly independent, but  $m = 2$ . Thus by the assertion (0), the map-germ  $\pi \circ f$  is diffeomorphic to a fold. Moreover we see  $f$  is an opening of  $\pi \circ f$  because  $f$  is frontal. The condition that  $\tilde{f}$  is an immersion is equivalent, in this case, to that  $f$  is a versal opening of  $\pi \circ f$  (§6 of [19]). In fact, up to diffeomorphism equivalence, let

$$f(u, w) = (u, \frac{1}{2}w^2, h_1(u, w), \dots, h_r(u, w)),$$

( $m = 2 + r$ ) and

$$dh_i(u, w) = a_i(u, w)du + b_i(u, w)d\left(\frac{1}{2}w^2\right) = a_i(u, w)du + wb_i(u, w)dw,$$

for some functions  $a_i, b_i, 1 \leq i \leq r$ . Then we have

$$\frac{\partial f}{\partial u} = (1, 0, a_1, \dots, a_r), \quad \frac{\partial f}{\partial w} = (0, w, wb_1, \dots, wb_r)$$

and the pair  $V_1 = \frac{\partial f}{\partial u}, V_2 = \frac{1}{w} \frac{\partial f}{\partial w} = (0, 1, b_1, \dots, b_r)$  gives a frame of the frontal  $f$ . Moreover  $\eta = \frac{\partial}{\partial w}$  gives the kernel field of  $f_*$  along  $\{w = 0\}$ . For any connection  $\nabla$ , we have the characteristic vector field  $\nabla_\eta^f V_2(u, 0) = (0, 0, \frac{\partial b_1}{\partial w}, \dots, \frac{\partial b_r}{\partial w})$ . Then  $\psi(0) \neq 0$  if and only if  $V_1(0, 0), V_2(0, 0), (\nabla_\eta^f V_2)(0, 0)$  are linearly independent. The condition is equivalent to that  $h_1(0, w), \dots, h_r(0, w)$  generate  $m_1^3/m_1^4$  over  $\mathbf{R}$  and it is equivalent to that  $f$  is a versal opening of the Whitney's cusp. Here  $m_1$  means the ideal consisting of function-germs  $h(w)$  with  $h(0) = 0$ . Then we see that  $f$  is diffeomorphic to cuspidal edge (see Proposition 6.8 (3)  $\ell = 2$  of [19]). The assertion (2) follows from Theorem 1.4 of [12].  $\square$

## 6 Swallowtail and open swallowtail

Based on results in [25] and [19], we summarize the characterization results on openings of the Whitney's cusp map-germ:

**Theorem 6.1** *Let  $f : (\mathbf{R}^2, p) \rightarrow M^m, m \geq 2$  be a germ of frontal with a non-degenerate singular point at  $p$ ,  $V_1, V_2 : (\mathbf{R}^2, p) \rightarrow TM$  an associated frame with  $\tilde{f}$  with  $V_2(p) \notin f_*(T_p \mathbf{R}^2)$ , and  $\eta : (\mathbf{R}^2, p) \rightarrow T\mathbf{R}^2$  an extension of a kernel field along of  $f_*$ . Let  $c : (\mathbf{R}, t_0) \rightarrow (\mathbf{R}^2, p)$  be a parametrization of the singular locus of  $f$ . Set  $\gamma = f \circ c : (\mathbf{R}, t_0) \rightarrow M$ . Suppose  $(\nabla \gamma)(t_0) = 0$  and  $(\nabla^2 \gamma)(t_0) \neq 0$ . Then  $f$  is diffeomorphic to an opening of Whitney's cusp, namely to the germ  $(u, t) \mapsto (u, t^3 + ut)$ . Moreover we have*

- (0) *Let  $m = 2$ . Then  $f$  is diffeomorphic to Whitney's cusp.*
- (1) *Let  $m = 3$ . Then  $f$  is diffeomorphic to the swallowtail if and only if*

$$V_1(c(t_0)), V_2(c(t_0)), (\nabla_\eta^f V_2)(c(t_0))$$

*are linearly independent in  $T_{f(p)}M$ .*

- (2) *Let  $m \geq 4$ . Then  $f$  is diffeomorphic to the open swallowtail if and only if*

$$(V_1 \circ c)(t_0), (V_2 \circ c)(t_0), ((\nabla_\eta^f V_2) \circ c)(t_0), (\nabla_{\partial/\partial t}^\gamma ((\nabla_\eta^f V_2) \circ c))(t_0)$$

*are linearly independent in  $T_{f(p)}M$ .*

*Proof:* The assertion (0) follows from Whitney's theorem (also see [40][35][34]). (1) follows from Proposition 1.3 of [25]. In general cases  $m \geq 2$ , we see that there exists a submersion  $\pi : (M, f(p)) \rightarrow (\mathbf{R}^2, 0)$  such that  $\pi^{-1}(0)$  is transverse to  $\tilde{f}(0) \subset T_{f(p)}M$ ,  $\pi \circ f$  satisfies the same condition with  $f$ , namely, that  $\pi \circ f$  is a frontal with the non-degenerate singular point at  $p$  and with the same singular locus with  $f$  and  $\eta(c(t_0))$  and  $c'(t_0)$  are linearly independent, but  $m = 2$ . Thus by the assertion (0), the map-germ  $\pi \circ f$  is diffeomorphic to the Whitney's cusp. Moreover we see  $f$  is an opening of Whitney's cusp because  $f$  is frontal.

Let  $f(u, t) = (u, t^3 + ut, h_1(u, t), \dots, h_r(u, t))$ ,  $m = 2 + r$  and  $dh_i = a_i du + b_i d(t^3 + ut) = (a_i + tb_i)du + (3t^2 + u)b_i dt$ , for some functions  $a_i = a_i(u, t)$ ,  $b_i = b_i(u, t)$ ,  $1 \leq i \leq r$ . Then we have

$$\frac{\partial f}{\partial u} = (1, t, a_1 + tb_1, \dots, a_r + tb_r), \quad \frac{\partial f}{\partial t} = (0, 3t^2 + u, (3t^2 + u)b_1, \dots, (3t^2 + u)b_r),$$

a frame  $V_1 = \frac{\partial f}{\partial u}$ ,  $V_2 = \frac{1}{3t^2 + u} \frac{\partial f}{\partial t} = (0, 1, b_1, \dots, b_r)$  of the frontal  $f$ , and a kernel field  $\eta = \frac{\partial}{\partial t}$  of  $f_*$ . We have

$$V_1(0, 0) = (1, 0, a_1(0, 0), \dots, a_r(0, 0)), \quad V_2(0, 0) = (0, 1, b_1(0, 0), \dots, b_r(0, 0)),$$

$$(\nabla_\eta^f V_2)(0, 0) = (0, 0, \frac{\partial b_1}{\partial t}(0, 0), \dots, \frac{\partial b_r}{\partial t}(0, 0)).$$

Let  $c(t) = (-3t^2, t)$ . Then  $\gamma(t) = f(c(t)) = (-3t^2, -2t^3, h_1(-3t^2, t), \dots, h_r(-3t^2, t))$  and

$$\nabla_\eta^f V_2(c(t)) = (0, 0, \frac{\partial b_1}{\partial t}(c(t)), \dots, \frac{\partial b_r}{\partial t}(c(t))).$$

Then we have

$$\nabla_{\partial/\partial t}^\gamma ((\nabla_\eta^f V_2) \circ c)|_{t=0} = (0, 0, \frac{\partial^2 b_1}{\partial t^2}(0, 0), \dots, \frac{\partial^2 b_r}{\partial t^2}(0, 0)).$$

Thus the condition of (2) is equivalent, in our case, to that  $f$  is a versal opening of  $\pi \circ f$  and then we see  $f$  is diffeomorphic to the open swallowtail (see Proposition 6.8 (3)  $\ell = 3$  of [19]). Thus we have the characterization (2).  $\square$

## 7 Characteristic vector field

Let  $\gamma : I \rightarrow M$  be an immersion. We set  $f = \nabla\text{-Tan}(\gamma)$  and suppose  $\nabla\gamma, \nabla^2\gamma$  are linearly independent at  $t = t_0$ . Note that  $\eta = \frac{\partial}{\partial t} - \frac{\partial}{\partial s}$  generates the field of kernels  $\text{Ker}(f_*)$  of the differential  $f_*$  along  $s = 0$ . Let  $F(t, s) = \frac{1}{s}(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s})$  as in the proof of Lemma 3.1. Then we have

$$(\nabla_\eta^f F)(t, s) = \{(\frac{\partial}{\partial t} - \frac{\partial}{\partial s})F^\lambda + (\Gamma_{\mu\nu}^\lambda(f(t, s))((\frac{\partial}{\partial t} - \frac{\partial}{\partial s})f^\mu)F^\nu\} \frac{\partial}{\partial x^\lambda}(f(t, s)).$$

Therefore we have

$$(\nabla_\eta^f F)(t, 0) = (\frac{\partial F}{\partial t} - \frac{\partial F}{\partial s})(t, 0).$$

We call the vector field  $(\nabla_\eta^f F)(t, 0)$  along  $\gamma$  the *characteristic vector field* of  $\nabla\text{-Tan}(\gamma)$ .

By straightforward calculations we have

**Lemma 7.1** *The characteristic vector field of  $\nabla\text{-Tan}(\gamma)$  is given by*

$$(\nabla_\eta^f F)^\lambda(t, 0) = (\gamma''')^\lambda + (\Gamma_{\mu\nu, \kappa}^\lambda + \frac{1}{2}\Gamma_{\rho\mu}^\lambda \Gamma_{\nu\kappa}^\rho + \frac{1}{2}\Gamma_{\mu\rho}^\lambda \Gamma_{\nu\kappa}^\rho)(\gamma')^\mu (\gamma')^\nu (\gamma')^\kappa + \frac{3}{2}(\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda)(\gamma')^\mu (\gamma'')^\nu.$$

**Lemma 7.2**  $(\nabla_\eta^f F)(t, 0) = (\nabla^3\gamma)(t)$  if the affine connection  $\nabla$  is torsion free.

*Proof:* We compare Lemma 7.1 and Lemma 2.4. The equality  $\frac{3}{2}(\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda) = 2\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda$  holds if and only if  $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$ . Then the equality  $\frac{1}{2}\Gamma_{\rho\mu}^\lambda \Gamma_{\nu\kappa}^\rho + \frac{1}{2}\Gamma_{\mu\rho}^\lambda \Gamma_{\nu\kappa}^\rho = \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\kappa}^\rho$  holds.  $\square$

Now let  $\gamma : I \rightarrow M$  be a directed curve and  $\gamma'(t_0) = 0$  for a  $t_0 \in I$ . Suppose  $c(t)u(t) = \gamma'(t)$  in a neighborhood of  $t_0$  for some frame  $u(t), u(t_0) \neq 0$  and a function  $c(t)$ . (In Lemma 4.2, we can take  $c(t) = k(t - t_0)^{k-1}$ ). We have defined in §4

$$f(t, s) = \nabla\text{-Tan}(\gamma)(t, s) := \varphi(\gamma(t), u(t), s),$$

using  $\nabla$ -geodesics  $\varphi(x, v, s)$ . Then  $V_1 = \frac{\partial f}{\partial s}$  and

$$V_2 = F(t, s) = \frac{1}{s} \left( \frac{\partial f}{\partial t}(t, s) - c(t) \frac{\partial f}{\partial s}(t, s) \right)$$

form a frame of the integral lifting of  $f$ . Let  $\eta = \frac{\partial}{\partial t} - c(t) \frac{\partial}{\partial s}$ , the kernel field. Then we have

**Lemma 7.3** *If  $\nabla$  is torsion free, then*

$$(\nabla_\eta^f F)(t, 0) = ((\nabla_{\partial/\partial t}^\gamma)^2 u)(t).$$

*Proof:* By straightforward calculations, we have

$$\begin{aligned} (\nabla_\eta^f F)^\lambda(t, 0) &= (u'')^\lambda + c' \Gamma_{\mu\nu}^\lambda u^\mu u^\nu + \frac{3}{2} c (\Gamma_{\mu\nu}^\lambda + \Gamma_{\nu\mu}^\lambda) u^\nu (u')^\mu \\ &\quad + \frac{1}{2} c^2 (2\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\rho\mu}^\lambda \Gamma_{\kappa\nu}^\rho + \Gamma_{\mu\rho}^\lambda \Gamma_{\kappa\nu}^\rho) u^\mu u^\nu u'^\kappa \\ &= (u'')^\lambda + \Gamma_{\mu\nu}^\lambda (\gamma'')^\mu u^\nu + \left( \frac{1}{2} \Gamma_{\nu\mu}^\lambda + \frac{3}{2} \Gamma_{\mu\nu}^\lambda \right) (\gamma')^\mu (u')^\nu \\ &\quad + \frac{1}{2} (2\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\rho\mu}^\lambda \Gamma_{\kappa\nu}^\rho + \Gamma_{\mu\rho}^\lambda \Gamma_{\kappa\nu}^\rho) (\gamma')^\mu (\gamma')^\kappa u'^\nu. \end{aligned}$$

On the other hand, we have

$$(\nabla_{\partial/\partial t}^\gamma u)^\lambda = (u')^\lambda + \Gamma_{\mu\nu}^\lambda (\gamma')^\mu u'^\nu,$$

and then

$$((\nabla_{\partial/\partial t}^\gamma)^2 u)^\lambda = (u'')^\lambda + \Gamma_{\mu\nu}^\lambda (\gamma'')^\mu u'^\nu + 2\Gamma_{\mu\nu}^\lambda (\gamma')^\mu (u')^\nu + (\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\mu\rho}^\lambda \Gamma_{\kappa\nu}^\rho) (\gamma')^\mu (\gamma')^\kappa u'^\nu.$$

Since  $\nabla$  is torsion free, we have  $\frac{1}{2}\Gamma_{\nu\mu}^\lambda + \frac{3}{2}\Gamma_{\mu\nu}^\lambda = 2\Gamma_{\mu\nu}^\lambda$  and  $\frac{1}{2}(2\Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\rho\mu}^\lambda \Gamma_{\kappa\nu}^\rho + \Gamma_{\mu\rho}^\lambda \Gamma_{\kappa\nu}^\rho) = \Gamma_{\mu\nu, \kappa}^\lambda + \Gamma_{\mu\rho}^\lambda \Gamma_{\kappa\nu}^\rho$ . Thus we have  $(\nabla_\eta^f F)(t, 0) = ((\nabla_{\partial/\partial t}^\gamma)^2 u)(t)$ .  $\square$

## 8 Euclidean case

Now we show Theorem 1.3 in Euclidean case, using the characterization results of singularities prepared in previous sections.

Let  $M = E^m$  be the Euclidean  $m$ -space with the connection  $\Gamma_{\mu\nu}^\lambda = 0$ . Then  $\varphi(x, v, s) = x + s v$ , that is  $h(x, v, s) = 0$ . If  $\gamma'(t_0) \neq 0$ , then the tangent surface is given by  $f(t, s) = \nabla\text{-Tan}(\gamma)(t, s) = \gamma(t) + s\gamma'(t)$  near  $t_0 \times \mathbf{R}$ . Then  $\frac{\partial f}{\partial t} = \gamma'(t) + s\gamma''(t)$ ,  $\frac{\partial f}{\partial s} = \gamma'(t)$  and we have

$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s} = s\gamma''(t)$ .  $F(t, s) = \frac{1}{s}(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}) = \gamma''(t)$ . Moreover we have  $(\nabla_{\partial/\partial t - \partial/\partial s}^f F)(t, 0) = \gamma'''(t)$ . Suppose  $\gamma'(t_0)$  and  $\gamma''(t_0)$  are linearly independent. By Lemma 3.1,  $f$  is a frontal with non-degenerate singular point at  $(t_0, 0)$ . We apply Theorem 5.1 to this situation. We take  $\eta = \frac{\partial}{\partial t} - \frac{\partial}{\partial s}$  and  $c(t) = (t, 0)$ . Then  $c'(t_0)$  and  $\eta(c(t_0))$  are linearly independent, that is,  $f_*(c'(t_0)) \neq 0$ . Take a coframe, namely, a system of germs of 1-forms  $L_i = L_i(t, s) : (I \times \mathbf{R}, (t_0, 0)) \rightarrow T^*E^m \setminus \zeta$  along  $f$ ,  $i = 1, \dots, m-2$ , which satisfy that

$$\langle L_i(t, s), \gamma'(t) \rangle = 0, \quad \langle L_i(t, s), \gamma''(t) \rangle = 0, \quad (i = 1, \dots, m-2),$$

and  $L_1(t, s), \dots, L_{m-2}(t, s)$  are linearly independent for  $(t, s) \in (I \times \mathbf{R}, (t_0, 0))$ . Here  $\zeta$  means the zero section. Actually we can take  $L$  to be independent of  $s$  in this case. Set  $\ell_i(t) = L_i(t, 0)$  and set  $\psi_i(t) = \langle \ell_i(t), \gamma'''(t) \rangle$  and

$$\psi(t) = (\psi_1(t), \dots, \psi_{m-2}(t)) = (\langle \ell_1(t), \gamma'''(t) \rangle, \dots, \langle \ell_{m-2}(t), \gamma'''(t) \rangle),$$

the characteristic function. Then we have that  $\psi(t_0) = 0$  if and only if  $\gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)$  are linearly dependent. We have

$$\langle \ell_i(t), \gamma'(t) \rangle = 0, \quad \langle \ell_i(t), \gamma''(t) \rangle = 0, \quad (i = 1, \dots, m-2),$$

and

$$\begin{aligned} 0 &= \langle \ell'_i(t), \gamma'(t) \rangle + \langle \ell_i(t), \gamma''(t) \rangle = \langle \ell'_i(t), \gamma'(t) \rangle, \\ 0 &= \langle \ell'_i(t), \gamma''(t) \rangle + \psi_i(t). \end{aligned}$$

Suppose  $\psi(t_0) = 0$ . Then  $\langle \ell'_i(t_0), \gamma''(t_0) \rangle = 0$  for any  $i$ , ( $1 \leq i \leq m-2$ ). Since we have also  $\langle \ell'_i(t_0), \gamma'(t_0) \rangle = 0$ , we obtain  $\langle \ell'_i(t_0), \gamma'''(t_0) \rangle = 0$ , because  $\gamma'(t_0)$  and  $\gamma''(t_0)$  are linearly independent and  $\gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)$  are linearly dependent. Now we have

$$\psi'_i(t) = \langle \ell'_i(t), \gamma'''(t) \rangle + \langle \ell_i(t), \gamma^{(4)}(t) \rangle.$$

Therefore we have  $\psi'_i(t_0) = \langle \ell_i(t_0), \gamma^{(4)}(t_0) \rangle$ . Thus under the condition  $\psi(t_0) = 0$ , we have  $\psi'(t_0) \neq 0$  if and only if  $\gamma'(t_0), \gamma''(t_0), \gamma^{(4)}(t_0)$  are linearly independent.

Thus Theorem 5.1 and Theorem 5.4 (1) imply Theorem 1.3 (1)(3) in the Euclidean case.

Suppose now  $\gamma'(t_0) = 0, \gamma''(t_0) \neq 0$ . Let  $c(t)u(t) = \gamma'(t)$  with  $u(t_0) \neq 0$ . Then  $c(t_0) = 0, c'(t_0) \neq 0$ . The tangent surface is defined by  $f(t, s) = \gamma(t) + su(t)$ . Then  $f$  is a frontal with a frame  $V_1(t, s) = \frac{\partial f}{\partial s}(t, s) = u(t)$  and  $V_2(t, s) = F(t, s) = \frac{1}{s}(\frac{\partial f}{\partial t} - c \frac{\partial f}{\partial s}) = u'(t)$ . Moreover  $\nabla_{\eta}^f F(t, 0) = u''(t)$ .

Now  $u(t_0), u'(t_0), u''(t_0)$  are linearly independent if and only if  $\gamma''(t_0), \gamma'''(t_0), \gamma^{(4)}(t_0)$  are linearly independent. Then in the case  $m = 3$ , by Theorem 6.1 (1), we have Theorem 1.3 (2). Let  $m \geq 4$ . Then we have  $\nabla_{\partial/\partial t}^{\gamma} \nabla_{\eta}^f F(t, 0) = u'''(t)$ . Now  $u(t_0), u'(t_0), u''(t_0), u'''(t_0)$  are linearly independent if and only if  $\gamma''(t_0), \gamma'''(t_0), \gamma^{(4)}(t_0), \gamma^{(5)}(t_0)$  are linearly independent. By Theorem 6.1 (2), we have Theorem 1.3 (4).

## 9 Proof of the characterization theorem

*Proof of Theorem 1.3 (1)(3) in the general torsion free case:*

Suppose  $(\nabla\gamma)(t_0)$  and  $(\nabla^2\gamma)(t_0)$  are linearly independent. Then  $f = \nabla\text{-Tan}(\gamma)$  is a frontal with

the frame  $V_1(t, s) = \frac{\partial f}{\partial t}(t, s)$  and  $V_2(t, s) = F(t, s)$  for the integral lifting  $\tilde{f}$ . We take coframe  $L = (L_1, \dots, L_{m-2})$ ,  $L_i : (I \times \mathbf{R}, (t_0, 0)) \rightarrow T^*M \setminus \zeta$  along  $f$  satisfying

$$\langle L_i(t, s), V_1(t, s) \rangle = 0, \quad \langle L_i(t, s), V_2(t, s) \rangle = 0, \quad (1 \leq i \leq m-2).$$

Set  $\ell_i(t) = L_i(t, 0)$ . Then we have

$$\langle \ell_i(t), (\nabla \gamma)(t) \rangle = 0, \quad \langle \ell_i(t), (\nabla^2 \gamma)(t) \rangle = 0.$$

Set

$$\psi_i(t) := \langle \ell_i(t), (\nabla_{\eta}^f F)(t, 0) \rangle, \quad (1 \leq i \leq m-2).$$

Since  $\nabla$  is torsion free, by Lemma 7.2, we have  $(\nabla_{\eta}^f F)(t, 0) = (\nabla^3 \gamma)(t)$  and so

$$\psi_i(t) = \langle \ell_i(t), (\nabla^3 \gamma)(t) \rangle.$$

Define the vector valued function

$$\psi : I \rightarrow \mathbf{R}^{m-2}, \quad \psi(t) = (\psi_1(t), \dots, \psi_{m-2}(t)).$$

Then  $\psi(t_0) = 0$  if and only if  $(\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), \nabla^3 \gamma(t_0)$  are linearly dependent.

Note that the covariant derivative  $(\nabla_{\frac{\partial}{\partial t}}^{\gamma} \psi_i)(t)$  of the function  $\psi_i(t)$  is equal to the ordinary derivative  $\psi'_i(t)$ . So we have

$$\psi'_i(t) = (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \psi_i)(t) = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t), (\nabla^3 \gamma)(t) \rangle + \langle \ell_i(t), (\nabla^4 \gamma)(t) \rangle.$$

Since  $\langle \ell_i(t), (\nabla \gamma)(t) \rangle = \langle \ell_i(t), (\nabla^2 \gamma)(t) \rangle = 0$ , we have

$$0 = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t), (\nabla \gamma)(t) \rangle + \langle \ell_i(t), (\nabla^2 \gamma)(t) \rangle = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t), (\nabla \gamma)(t) \rangle.$$

If  $\psi(t_0) = 0$  namely if  $(\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), (\nabla^3 \gamma)(t_0)$  are linearly dependent, then we have that  $\langle \ell_i(t_0), (\nabla^3 \gamma)(t_0) \rangle = 0, (1 \leq i \leq m-2)$ , since  $(\nabla^3 \gamma)(t_0)$  is a linear combination of  $(\nabla \gamma)(t_0)$  and  $(\nabla^2 \gamma)(t_0)$ . Thus we have

$$0 = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t_0), (\nabla^2 \gamma)(t_0) \rangle + \langle \ell_i(t_0), (\nabla^3 \gamma)(t_0) \rangle = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t_0), (\nabla^2 \gamma)(t_0) \rangle.$$

Moreover we have  $\langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t_0), (\nabla^3 \gamma)(t_0) \rangle = 0$ . Therefore we have

$$\psi'_i(t_0) = \langle (\nabla_{\frac{\partial}{\partial t}}^{\gamma} \ell_i)(t_0), (\nabla^3 \gamma)(t_0) \rangle + \langle \ell_i(t_0), (\nabla^4 \gamma)(t_0) \rangle = \langle \ell_i(t_0), (\nabla^4 \gamma)(t_0) \rangle.$$

Therefore, if  $\psi(t_0) = 0$  and  $(\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), (\nabla^4 \gamma)(t_0)$  are linearly independent, then  $\psi'(t_0) \neq 0$ .

Now, by Theorem 5.1, we see that, if  $\dim(M) = m = 3$ , and  $\psi(t_0) = 0, \psi'(t_0) \neq 0$ , then  $f$  is diffeomorphic to the folded umbrella (cuspidal cross cap) at  $(t_0, 0)$ . Moreover, if  $\dim(M) \geq 3$  and  $\psi(t_0) \neq 0$ , then  $f$  is diffeomorphic to the cuspidal edge at  $(t_0, 0)$ .  $\square$

*Proof of Theorem 1.3 (2)(4) in the general torsion free case:*

Let  $\gamma : I \rightarrow M$  is not an immersion at  $t_0$ ,  $\gamma'(t_0) = 0$ , but  $\gamma''(t_0) \neq 0$ . Let  $c(t)u(t) = \gamma'(t), u(t_0) \neq$

0. Then the  $\nabla$ -tangent surface is defined by  $f(t, s) = \varphi(\gamma(t), u(t), s)$  using the geodesics  $\varphi(x, v, s)$  on  $TM$ . Then we have the frame

$$V_1(t, s) = \frac{\partial f}{\partial s}(t, s), \quad V_2(t, s) = F(t, s) = \frac{1}{s} \left( \frac{\partial f}{\partial t} - c(t) \frac{\partial f}{\partial s} \right).$$

We set  $\eta = \frac{\partial}{\partial t} - c(t) \frac{\partial}{\partial s}$ . Then, by Lemma 7.3, we have  $(\nabla_\eta^f F)(t, 0) = (\nabla_{\partial/\partial t}^\gamma {}^2 u)(t)$ . Therefore we have  $\nabla_{\partial/\partial t}^\gamma ((\nabla_\eta^f F)(t, 0)) = (\nabla_{\partial/\partial t}^\gamma {}^3 u)(t)$ . Now, by Lemma 4.2,  $V_1(t_0, 0), V_2(t_0, 0), (\nabla_\eta^f F)(t_0, 0)$  are linearly independent if and only if  $u(t_0), (\nabla_{\partial/\partial t}^\gamma u)(t_0), (\nabla_{\partial/\partial t}^\gamma {}^2 u)(t_0)$  are linearly independent, and the condition is equivalent to that  $(\nabla^2 \gamma)(t_0), (\nabla^3 \gamma)(t_0), (\nabla^4 \gamma)(t_0)$  are linearly independent. Then in the case  $m = 3$ , by Theorem 6.1 (1), we have Theorem 1.3 (2).

Let  $m \geq 4$ . Then  $V_1(t_0, 0), V_2(t_0, 0), (\nabla_\eta^f F)(t_0, 0), \nabla_{\partial/\partial t}^\gamma ((\nabla_\eta^f F)(t, 0))|_{t=t_0}$  are linearly independent if and only if  $u(t_0), (\nabla_{\partial/\partial t}^\gamma u)(t_0), (\nabla_{\partial/\partial t}^\gamma {}^2 u)(t_0), (\nabla_{\partial/\partial t}^\gamma {}^3 u)(t_0)$  are linearly independent, and the condition is equivalent to that  $(\nabla^2 \gamma)(t_0), (\nabla^3 \gamma)(t_0), (\nabla^4 \gamma)(t_0), (\nabla^5 \gamma)(t_0)$  are linearly independent. By Theorem 6.1 (2), we have Theorem 1.3 (4).

## 10 Proof of genericity theorem 1

In general, let  $\gamma : I \rightarrow M$  be a  $C^\infty$  curve and  $t_0 \in I$ . Define

$$a_1 := \inf \left\{ k \mid k \geq 1, (\nabla^k \gamma)(t_0) \neq 0 \right\}.$$

Note that  $\gamma$  is an immersion at  $t_0$  if and only if  $a_1 = 1$ . If  $a_1 < \infty$ , then define

$$a_2 := \inf \left\{ k \mid \text{rank} \left( (\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), \dots, (\nabla^k \gamma)(t_0) \right) = 2 \right\}.$$

We have  $1 \leq a_1 < a_2$ . If  $a_i < \infty, 1 \leq i < \ell \leq m$ , then define  $a_\ell$  inductively by

$$a_\ell := \inf \left\{ k \mid \text{rank} \left( (\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), \dots, (\nabla^k \gamma)(t_0) \right) = \ell \right\}.$$

If  $a_m < \infty$ , then we call the strictly increasing sequence  $(a_1, a_2, \dots, a_m)$  of natural numbers the  $\nabla$ -type of  $\gamma$  at  $t_0$ .

To obtain our Theorem 1.1, we show

**Proposition 10.1** *Let  $(M, \nabla)$  be a manifold of dimension  $m$  with an affine connection  $\nabla$ . Then there exists an open dense set  $\mathcal{U}$  in the set of  $C^\infty$  curves from  $I$  to  $M$  in Whitney  $C^\infty$  topology, such that for any  $\gamma$  belonging to  $\mathcal{U}$  and for any  $t_0 \in I$ ,  $\gamma$  is of  $\nabla$ -type  $(1, 2, 3)$  or  $(1, 2, 4)$  if  $m = 3$ , and  $(1, 2, 3, \dots, m-1, m)$  or  $(1, 2, 3, \dots, m-1, m+1)$  if  $m \geq 4$ , at  $t_0$ .*

*Proof of Propositions 10.1:* First we remark that, for any local coordinates on  $M$ , and for each  $k = 1, 2, \dots$ , the iterated covariant derivative  $(\nabla^k \gamma)(t)$  is expressed as  $(\nabla^k \gamma)(t) = \gamma^{(k)}(t) + P$ , by a polynomial  $P$  of  $\gamma^{(i)}(t), 0 \leq i < k$  and  $(\partial^\alpha \Gamma_{\mu\nu}^\lambda / \partial x^\alpha)(\gamma(t)), |\alpha| \leq k-2$  (cf. Lemma 2.4). Therefore, for positive integer  $r$ , there exists an algebraic diffeomorphism of  $\Phi : J^r(I, M)_{t_0, q} \rightarrow J^r(I, M)_{t_0, q}$  of the  $r$ -jet space  $J^r(I, M)_{t_0, q} = \{j^r \gamma(t_0) \mid \gamma : (I, t_0) \rightarrow (M, q)\}$  satisfying the following conditions: if  $\Phi(j^r \gamma(t_0)) = j^r \beta(t_0), \beta : (I, t_0) \rightarrow (M, q)$ , then  $(\nabla^k \gamma)(t_0) = \beta^{(k)}(t_0), 1 \leq k \leq r$ . In particular, for any  $1 \leq \ell \leq r$ , and for any  $1 \leq i_1 < i_2 < \dots < i_\ell \leq r$  and for any

$j, 0 \leq j \leq m$ ,  $\text{rank}((\nabla^{i_1}\gamma)(t_0), \dots, (\nabla^{i_\ell}\gamma)(t_0)) = j$  if and only if  $\text{rank}(\beta^{(i_1)}(t_0), \dots, \beta^{(i_\ell)}(t_0)) = j$ . Let  $r \geq m + 1$ . We set

$$S_\nabla := \{j^r\gamma(t_0) \mid (\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), \dots, (\nabla^m\gamma)(t_0) \text{ are linearly dependent and} \\ (\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), \dots, (\nabla^{m-1}\gamma)(t_0), (\nabla^{m+1}\gamma)(t_0) \text{ are linearly dependent.}\}$$

is an algebraic set of codimension 2. The above diffeomorphism  $\Phi$  maps  $S_\nabla$  to  $S_{\nabla_0}$ . Here  $\nabla_0$  is the trivial connection on  $\mathbf{R}^m$ . Thus the calculation is reduced to the trivial case which is well known (see [14][19]). Note that  $S_\nabla$  is intrinsically defined by the given connection  $\nabla$ . Then we have the associated closed stratified subbundle  $S_\nabla(I, M)$  of the  $r$ -jet bundle  $J^r(I, M) \rightarrow I \times M$  of codimension 2. By the transversality theorem

$$\mathcal{U} := \{\gamma : I \rightarrow M \mid j^r\gamma : I \rightarrow J^r(I, M) \text{ is transverse to } S_\nabla(I, M)\}$$

is open dense in Whitney  $C^\infty$  topology (see [14]). Let  $\gamma \in \mathcal{U}$  and  $t_0 \in I$ . Since  $S_\nabla(I, M)$  is codimension 2,  $j^r\gamma(t_0) \notin S_\nabla(I, M)$ . This means that

$$(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), \dots, (\nabla^m\gamma)(t_0)$$

are linearly independent, or, they are linearly dependent but

$$(\nabla\gamma)(t_0), (\nabla^2\gamma)(t_0), \dots, (\nabla^{m-1}\gamma)(t_0), (\nabla^{m+1}\gamma)(t_0)$$

are linearly independent. In the first case,  $\gamma$  is of  $\nabla$ -type  $(1, 2, \dots, m-1, m)$ . In the second case,  $\gamma$  is of type  $\nabla$ -type  $(1, 2, \dots, m-1, m+1)$ . Thus we have Proposition 10.1.  $\square$

**Remark 10.2** Let  $\mathbf{a} = (a_1, a_2, \dots, a_m)$  be any strictly increasing sequence of positive integers. For an integer  $r \geq a_m$ , we define, in the  $r$ -jet bundle  $J^r(I, M)$ ,

$$\Sigma_{\mathbf{a}}(I, M) := \{j^r\gamma(t_0) \in J^r(I, M) \mid \gamma \text{ is of } \nabla\text{-type } \mathbf{a}\}.$$

Then  $\Sigma_{\mathbf{a}}(I, M)$  is a stratified subbundle of  $J^r(I, M)$  over  $I \times M$  with an algebraic typical fiber. It can be shown, as in the proof of Propositions 10.1, that the codimension of  $\Sigma_{\mathbf{a}}(I, M)$  in  $J^r(I, M)$  is independent of  $\nabla$  and is given by  $\sum_{i=1}^m (a_i - i)$  (see [19] for the flat case).

*Proof of Theorem 1.1:* We may suppose  $\nabla$  is torsion free (Remark 2.3). Then Proposition 10.1, and Theorem 1.3 imply Theorem 1.1.

## 11 Perturbations of integral curves

To treat directed curves (see §4), we consider  $PTM = \text{Gr}(1, TM)$  with the natural projection  $\pi : PTM \rightarrow M$  and the tautological subbundle  $D \subset TPTM$  on the tangent bundle of  $PTM$ : For any  $(x, \ell) \in PTM$  and for any  $v \in T_{(x, \ell)}PTM$ ,  $v \in D_{(x, \ell)}$  if and only if  $\pi_*(v) \in \ell \subset T_x M$ . A curve  $\tilde{\gamma} : I \rightarrow PTM$  is called *integral* if  $\tilde{\gamma}_*(\partial/\partial t) \in D_{\tilde{\gamma}(t)}$ , for any  $t \in I$ . Recall that  $\gamma = \pi \circ \tilde{\gamma}$  with the lifting  $\tilde{\gamma}$  is called a *directed curve*. Then we have

**Proposition 11.1** *Let  $M$  be a manifold of dimension  $m$  with an affine connection  $\nabla$ . Then there exists an open dense subset  $\mathcal{O}$  in the space of  $C^\infty$  integral curves  $I \rightarrow PTM$  with Whitney  $C^\infty$  topology such that for any  $\tilde{\gamma} \in \mathcal{O}$  and for any  $t_0 \in I$ ,  $\gamma = \pi \circ \tilde{\gamma} : I \rightarrow M$  is of  $\nabla$ -type*

$$(1, 2, 3), (1, 2, 4), \text{ or } (2, 3, 4),$$

if  $m = \dim(M) = 3$ , and

$$(1, 2, 3, 4, \dots, m-1, m), (1, 2, 3, 4, \dots, m-1, m+1), \text{ or } (2, 3, 4, 5, \dots, m, m+1).$$

if  $m \geq 4$ , at  $t_0$ .

To treat directed curves, we introduce the following notion: Let  $u : I \rightarrow TM$  be a vector field along a curve  $\gamma : I \rightarrow M$ . For  $t_0 \in I$ , we set

$$b_i := \inf \left\{ k \mid \text{rank} \left( u(t_0), (\nabla_{\partial/\partial t}^\gamma u)(t_0), \dots, ((\nabla_{\partial/\partial t}^\gamma)^{k-1} u)(t_0) \right) = i \right\}.$$

We have  $1 \leq b_1 < b_2 < \dots < b_m$ , if each  $b_i < \infty$ . Then we call the strictly increasing sequence  $(b_1, b_2, \dots, b_m)$  of natural numbers the  $\nabla$ -type of  $u$  at  $t_0$ . Note that the  $\nabla$ -type of a curve  $\gamma$  which we have defined in above is the  $\nabla$ -type of the velocity vector field  $\gamma' : I \rightarrow TM$  along  $\gamma$ .

Let  $\gamma : (\mathbf{R}, t_0) \rightarrow M$  be a germ of directed curve with an integral lifting  $\tilde{\gamma} : (\mathbf{R}, t_0) \rightarrow PTM$  generated by a frame  $u : (\mathbf{R}, t_0) \rightarrow TM$ ,  $u(t_0) \neq 0$ . Then  $b_1 = 1$  for  $u$ , since  $u(t_0) \neq 0$ .

Generalizing Lemma 4.2 (3), we have:

**Lemma 11.2** *If  $\nabla$ -type of  $u$  is  $(1, b_2, \dots, b_m)$  and the order of  $c$  at  $t_0$  is  $\ell$ , that is,  $c(t_0) = \dots = c^{(\ell-1)}(t_0) = 0, c^{(\ell)}(t_0) \neq 0$ , then  $\gamma$  is of  $\nabla$ -type  $(a_1, a_2, \dots, a_m) = (1 + \ell, b_2 + \ell, \dots, b_m + \ell)$ .*

*Proof:* By taking covariant derivative  $\nabla$   $\ell$ -times of the both sides of  $c(t)u(t) = \gamma'(t)$ , we have  $(\nabla \gamma)(t_0) = \dots = (\nabla^\ell \gamma)(t_0) = 0, (\nabla^{\ell+1} \gamma)(t_0) \neq 0$ . Therefore  $a_1 = 1 + \ell$ . In general, we have inductively

$$\text{rank} \left( (\nabla \gamma)(t_0), (\nabla^2 \gamma)(t_0), \dots, (\nabla^k \gamma)(t_0) \right) = \text{rank} \left( u(t_0), (\nabla u)(t_0), \dots, (\nabla^{k-\ell-1} u)(t_0) \right),$$

for any  $k \geq 1 + \ell$ . Therefore we have  $a_i = b_i + \ell, 1 \leq i \leq m$ .  $\square$

We need also the following lemma on local perturbations of integral curves.

**Lemma 11.3** *Let  $a < t_1 < t_2 < b$  and  $\tilde{\gamma}, \tilde{\alpha} : (a, b) \rightarrow PTR^m$  be integral curves. Then there exists an integral curve  $\tilde{\beta} : (a, b) \rightarrow PTR^m$  such that  $\tilde{\beta}(t) = \tilde{\alpha}(t), a < t \leq t_1$  and  $\tilde{\beta}(t) = \tilde{\gamma}(t), t_2 \leq t < b$ . If  $\tilde{\alpha}$  is sufficiently close to  $\tilde{\gamma}$  on  $[t_1, t_2]$  in Whitney  $C^\infty$  topology, then  $\tilde{\beta}$  can be taken to be close to  $\tilde{\gamma}$  on  $(a, b)$  in Whitney  $C^\infty$  topology.*

*Proof:* Let  $x = (x^\lambda)$  be a system of coordinates of  $\mathbf{R}^m$  and  $(x, \xi) = (x^\lambda, \xi_\lambda)$  be the associated system of coordinates of  $T\mathbf{R}^m$ . Let  $(x \circ \tilde{\gamma})'(t) = c(t)u(t)$ ,  $(x \circ \tilde{\alpha})'(t) = e(t)v(t)$ , for some  $c, e : (a, b) \rightarrow \mathbf{R}$  and  $u, v : (a, b) \rightarrow \mathbf{R}^m \setminus \{0\}$ . Then we take a function  $f(t)$  and  $w : (a, b) \rightarrow \mathbf{R}^m$  such that  $f(t) = e(t)$  on  $(a, t_1]$ ,  $f(t) = c(t)$  on  $[t_2, b)$ ,  $w(t) = v(t)$  on  $(a, t_1]$ ,  $w(t) = u(t)$  on  $[t_2, b)$  and

$$\int_{t_1}^{t_2} f(t)w(t)dt = (x \circ \tilde{\gamma})(t_1) - (x \circ \tilde{\alpha})(t_1) + \int_{t_1}^{t_2} c(t)u(t)dt.$$

Then we have the required  $\tilde{\beta}$  by  $(\xi \circ \tilde{\beta})(t) = w(t)$  and

$$(x \circ \tilde{\beta})(t) = (x \circ \tilde{\alpha})(t_1) + \int_{t_1}^t f(t)w(t)dt, \quad (a < t < b).$$

$\square$

*Proof of Proposition 11.1:* Let  $\tilde{\gamma} : (\mathbf{R}, t_0) \rightarrow PTM$  be a germ of integral curve with  $\gamma = \pi \circ \tilde{\gamma}$ . Let  $c(t)u(t) = \gamma'(t)$  for some frame  $u : (\mathbf{R}, t_0) \rightarrow TM$  along  $\gamma$ ,  $u(t_0) \neq 0$ , and for some function  $c : (\mathbf{R}, t_0) \rightarrow \mathbf{R}$ . Note that  $\tilde{\gamma}$  is determined by the frame  $u$ . The frame  $u$  is determined up to the multiplication of functions  $b(t)$  with  $b(t_0) \neq 0$ . Given the initial point  $q = \gamma(t_0)$ , the pair  $(u, c)$  determines the directed curve  $\gamma$  uniquely. Moreover  $(\nabla^k u)(t_0) = u^{(k)}(t_0) + Q$ , by a polynomial  $Q$  of  $u^{(i)}(t_0), c^{(i)}(t_0), 0 \leq i < k$  and  $(\partial^\alpha \Gamma_{\mu\nu}^\lambda / \partial x^\alpha)(q), |\alpha| \leq k-2$ . In particular  $(\nabla^k u)(t_0)$  depends only on  $k$ -jet of  $(c, u)$  and just on the position  $q = \gamma(t_0)$ .

Let us consider the  $r$ -jet bundle  $J^r(I, \mathbf{R} \times (TM \setminus \zeta))$  over  $I \times \mathbf{R} \times (TM \setminus \zeta)$ , where  $\zeta$  is the zero-section. For the projection  $I \times \mathbf{R} \times (TM \setminus \zeta) \rightarrow I \times M$ , take the fiber  $J^r(I, \mathbf{R} \times (TM \setminus \zeta))_{t_0, q}$  over a  $(t_0, q) \in I \times M$ , and consider the set

$$\begin{aligned} S_\nabla &:= \{j^r(c, u)(t_0) \mid u(t_0), (\nabla u)(t_0), \dots, (\nabla^{m-1} u)(t_0) \text{ are linearly dependent} \\ &\quad \text{and } u(t_0), (\nabla u)(t_0), \dots, (\nabla^{m-2} u)(t_0), (\nabla^m u)(t_0) \text{ are linearly dependent}\}. \\ S'_\nabla &:= \{j^r(c, u)(t_0) \mid u(t_0), (\nabla u)(t_0), \dots, (\nabla^{m-1} u)(t_0) \text{ are linearly dependent} \\ &\quad \text{and } c(t_0) = 0\} \\ S''_\nabla &:= \{j^r(c, u)(t_0) \mid c(t_0) = c'(t_0) = 0\}. \end{aligned}$$

Then, for any but fixed system of local coordinates around  $q$  of  $M$ ,  $S_\nabla, S'_\nabla, S''_\nabla$  are algebraic sets of codimension  $\geq 2$ . Let  $S_\nabla(I, M), S'_\nabla(I, M), S''_\nabla(I, M)$  be the corresponding subbundle of  $J^r(I, \mathbf{R} \times (TM \setminus \zeta))$  over  $I \times M$ . For any subinterval  $J \subset I$ , we set

$$\begin{aligned} \tilde{\mathcal{O}}_J &:= \{(c, u) : I \rightarrow \mathbf{R} \times (TM \setminus \zeta) \mid j^r(c, u) : I \rightarrow J^r(I, \mathbf{R} \times (TM \setminus \zeta)) \\ &\quad \text{is transverse to } S_\nabla(I, M), S'_\nabla(I, M), S''_\nabla(I, M) \text{ over } J\}. \end{aligned}$$

Then  $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_I$  is open dense in Whitney  $C^\infty$  topology. Let  $(c, u) \in \tilde{\mathcal{O}}$  and  $t_0 \in I$ . Then  $j^r(c, u)(t_0) \notin S_\nabla \cup S'_\nabla \cup S''_\nabla$ . Since  $j^r(c, u)(t_0) \notin S_\nabla$ , we have that the  $\nabla$ -type of  $u$  is  $(1, 2, \dots, m-1, m)$  or  $(1, 2, \dots, m-1, m+1)$ . Since  $j^r(c, u)(t_0) \notin S'_\nabla$ , if  $c(t_0) = 0$  then  $\nabla$ -type of  $u$  must be  $(1, 2, \dots, m-1, m)$ . On the other hand, since  $j^r(c, u)(t_0) \notin S''_\nabla$ , we have that  $c(t_0) \neq 0$  or  $c(t_0) = 0, c'(t_0) \neq 0$ , i.e. the order of  $c$  at  $t_0$  is 0 or 1. We set

$$\mathcal{O}_J := \{\tilde{\gamma} : I \rightarrow PTM \text{ integral} \mid \exists (c, u) \in \tilde{\mathcal{O}}_J, \tilde{\gamma}(t) = \langle u(t) \rangle_{\mathbf{R}}, (\pi \circ \tilde{\gamma})'(t) = c(t)u(t)\}.$$

We will show, for any compact subinterval  $J \subset I$ , that  $\mathcal{O}_J$  is open dense and  $\mathcal{O} = \mathcal{O}_I$  is open dense in the space of integral curves with Whitney  $C^\infty$  topology.

That  $\mathcal{O}_J$  and  $\mathcal{O}$  are open is clear, since  $\tilde{\mathcal{O}}$  is open.

We will show  $\mathcal{O}_J$  is dense. Let  $\tilde{\gamma} : I \rightarrow PTM$  be any integral curve and  $\mathcal{I}$  be any open neighborhood of  $\tilde{\gamma}$ . Set  $\gamma = \pi \circ \tilde{\gamma} : I \rightarrow M$ . Take any frame  $u$  associated to  $\tilde{\gamma}$ . Then there exists uniquely  $c : I \rightarrow \mathbf{R}$  which satisfies  $c(t)u(t) = \gamma'(t), t \in I$ . Take a compact subinterval  $J' \subset I$  such that  $J \subsetneq J'$ . We approximate  $(c, u)$  by some  $(e, v) \in \tilde{\mathcal{O}}_J$  and that  $(e, v) = (c, u)$  outside of  $J'$ . Then  $v$  generates a curve  $\rho : I \rightarrow PTM, \rho(t) = \langle v(t) \rangle_{\mathbf{R}}$ , which approximates  $\tilde{\gamma}$ , however  $\rho$  may not be an integral curve. Consider the vector field  $(\frac{\partial}{\partial t}, ev)$  along the graph of  $\pi \circ \rho$  in  $I \times M$ . Extend  $(\frac{\partial}{\partial t}, v(t))$  to a vector field  $(\frac{\partial}{\partial t}, V(t, x))$  over  $I \times M$  with a support contained in  $I \times K$  for some compact  $K \subset M$ . Take  $t_0 \in J$ . Take the integral curve  $\alpha : I \rightarrow M$  of the vector field  $(\frac{\partial}{\partial t}, e(t)V(t, x))$  through  $(t_0, \alpha(t_0))$ . Then  $\alpha'(t) = e(t)V(t, \alpha(t))$ . Define the vector field  $w : I \rightarrow TM$  over  $\alpha$  by  $w(t) = V(t, \alpha(t))$ . Then we have  $\alpha'(t) = e(t)w(t)$ . If we choose  $(e, v)$  sufficiently close to  $(c, u)$ , then  $w(t) \neq 0$  and  $(e, w) \in \tilde{\mathcal{O}}_J$ . However the integral curve  $\tilde{\alpha}$  defined by  $w$  may not belong to  $\mathcal{I}$ , an open set for Whitney  $C^\infty$  topology. Further we modify the perturbation  $(e, v)$  over  $J' \setminus J$  and the extension  $V$  over  $(J' \setminus J) \times M$  to obtain an integral

curve  $\tilde{\beta}$  such that  $\tilde{\beta} = \tilde{\alpha}$  on  $J$  and  $\tilde{\beta} = \tilde{\gamma}$  outside of  $J'$ , using the method of Lemma 11.3. Then the integral curve  $\tilde{\beta}$  approximates  $\tilde{\gamma}$  and belongs to  $\mathcal{I}$ , while  $(e, w) \in \tilde{\mathcal{O}}_J$ . Thus we have seen that  $\mathcal{O}_J$  is dense, for any compact subinterval  $J \subset I$ .

Since  $\mathcal{O} = \bigcap_{J \subset I} \mathcal{O}_J$ , the intersection over compact subintervals  $J \subset I$ , we have that  $\mathcal{O}$  is residual, and therefore that  $\mathcal{O}$  is dense in Whitney  $C^\infty$  topology [14].

Thus we have that  $\mathcal{O}$  is open dense in Whitney  $C^\infty$  topology. Then, using Lemma 11.2, we have the required result.  $\square$

**Remark 11.4** By the same method as above, we have that the codimension of jets of integral curves such that the projections are of  $\nabla$ -type  $(a_1, a_2, \dots, a_m)$  is given by

$$\ell + \sum_{i=1}^m (b_i - i) = a_1 - 1 + \sum_{i=2}^m (a_i - a_1 - i + 1),$$

for any affine connection  $\nabla$ . Note that the codimension is calculated in Theorem 5.6 of [19] in the flat case (cf. Theorem 5.8, Theorem 3.3 of [19]).

*Proof of Theorem 1.2:* We may suppose  $\nabla$  is torsion free (Remark 2.3). Then Proposition 11.1 and Theorem 1.3 imply Theorem 1.2.  $\square$

## 12 Tangent surfaces to torsionless curves and fold singularities

Let  $(M, \nabla)$  be a manifold of dimension  $m \geq 3$  with a torsion free affine connection  $\nabla$ . Consider a curve  $\gamma : I \rightarrow M$  such that  $(\nabla\gamma)(t), (\nabla^2\gamma)(t)$  are linearly independent for any  $t \in I$ . Though the torsion is not defined in general, we can define:

**Definition 12.1** A curve  $\gamma : I \rightarrow M$  is called *torsionless* if  $(\nabla\gamma)(t), (\nabla^2\gamma)(t)$  are linearly independent but  $(\nabla\gamma)(t), (\nabla^2\gamma)(t), (\nabla^3\gamma)(t)$  are linearly dependent everywhere.

The situation is, by any means, non-generic. However to study torsionless curves is an interesting geometric problem (cf. [5]).

**Remark 12.2** Let  $M$  be a Riemannian manifold with the Levi-Civita connection  $\nabla$ . The notion of torsion is well-defined for the curve  $\gamma$  parametrized by the arc-length, if  $\nabla\gamma, \nabla^2\gamma$  are linearly independent. Then the condition that the torsion of  $\gamma$  is zero if and only if  $\gamma$  satisfies the third order non-linear ordinary differential equation,

$$\nabla^3\gamma - \frac{\|\nabla^2\gamma\|'}{\|\nabla^2\gamma\|} \nabla^2\gamma + \|\nabla^2\gamma\|^2 \nabla\gamma = 0.$$

Then in particular  $\gamma$  is torsionless in our sense.

If  $(M, \nabla)$  is projectively flat, namely, if it is projectively equivalent to the Euclidean space  $(E^m, \nabla_0)$  with the standard connection  $\nabla_0$ , then it is well-known that any torsionless curve is a “plane curve”, therefore its  $\nabla$ -tangent surface is “folded” into a totally geodesic surface. Note that a Riemannian manifold is locally projectively flat if and only if it has a constant curvature (Beltrami’s Theorem, see p.352 of [36] or p.97 of [9]).

A map-germ  $f : (\mathbf{R}^2, p) \rightarrow (M, f(p))$  is called a *fold*, or an *embedded fold*, if it is diffeomorphic to

$$(t, s) \mapsto (t + s, t^2 + 2st, 0, \dots, 0),$$

which is diffeomorphic also to  $(u, w) \mapsto (u, \frac{1}{2}w^2, 0, \dots, 0)$ . The fold singularities appear in other geometric problems also (see [11] for instance). For our problem, we have

**Proposition 12.3** *Let  $(M, \nabla)$  be locally projectively flat around  $q \in M$  and  $\gamma : (\mathbf{R}, t_0) \rightarrow (M, q)$  a germ of torsionless curve. Then the germ of  $\nabla$ -tangent surface  $\nabla\text{-Tan}(\gamma) : (\mathbf{R}^2, (t_0, 0)) \rightarrow (M, q)$  to  $\gamma$  is a fold. In particular it is a generically two-to-one mapping.*

*Proof:* There exists a germ of projective equivalence  $\varphi : (M, q) \rightarrow (E^m, 0)$  such that  $\varphi \circ \gamma : (\mathbf{R}, 0) \rightarrow (E^m, 0)$  is a strictly convex plane curve in  $E^2 \times \{0\} \subset E^m$ . Then, by Theorem 5.4, the tangent surface  $\nabla\text{-Tan}(\varphi \circ \gamma) = \varphi \circ (\nabla\text{-Tan}(\gamma))$  is diffeomorphic to a fold regarded as a map-germ  $(\mathbf{R}^2, (0, 0)) \rightarrow (\mathbf{R}^2, (0, 0))$ . Therefore  $\nabla\text{-Tan}(\gamma)$  is diffeomorphic to a fold, regarded as a surface-germ  $(\mathbf{R}^2, (t_0, 0)) \rightarrow (M, q)$ .  $\square$

A map-germ  $f : (\mathbf{R}^2, p) \rightarrow (M^3, f(p))$  to a three dimensional space is called a  $(2, 5)$ -cuspidal edge, if it is diffeomorphic to

$$(u, w) \mapsto (u, w^2, w^5).$$

Then  $f$  is a frontal with a non-degenerate singular point at  $p$ , and the characteristic function  $\psi$  vanishes identically (see §5). In fact, the model germ  $(u, w) \mapsto (u, w^2, w^5)$  (the “suspension” of  $(2, 5)$ -cusp) is a frontal with a non-degenerate singular point at 0, the kernel field  $\partial/\partial u$  for  $f_*$  is transverse to the singular curve  $w = 0$ , and the characteristic function  $\psi(u) \equiv 0$ . However  $f$  is injective and it is never diffeomorphic to a tangent surface to any torsionless curve in a projectively flat space, by Proposition 12.3. Nevertheless it can be a  $\nabla$ -tangent surface of a torsionless curve for a torsion free affine connection on  $M^3$ .

**Example 12.4** Let  $\nabla$  be the torsion free affine connection on  $\mathbf{R}^3$  with coordinates  $x_1, x_2, x_3$  defined by  $\Gamma_{\mu\nu}^\lambda = x_1 + x_2^2$  if  $(\lambda, \mu, \nu) = (3, 1, 2), (3, 2, 1)$  and otherwise  $\Gamma_{\mu\nu}^\lambda = 0$ . Let  $\gamma : \mathbf{R} \rightarrow \mathbf{R}^3$  be the immersion defined by  $\gamma(t) = (-t^2, t, 0)$ . Then  $\Gamma_{\mu\nu}^\lambda = 0$  along  $\gamma$ . We have  $(\nabla\gamma)(t) = (-2t, 1, 0)$ ,  $(\nabla^2\gamma)(t) = (-2, 0, 0)$ ,  $(\nabla^3\gamma)(t) = (0, 0, 0)$ . Therefore  $\gamma$  is torsionless. For any  $t_0 \in \mathbf{R}$ , the  $\nabla$ -geodesic with the initial condition  $(\gamma(t_0), \gamma'(t_0))$  is given by  $(-2t_0s - t_0^2, s + t_0, \frac{1}{3}t_0s^4)$ . Therefore we have

$$f(t, s) = \nabla\text{-Tan}(\gamma)(t, s) = (-2ts - t^2, s + t, \frac{1}{3}ts^4).$$

Set  $u = s + t, w = s$ . Then we see that  $f$  is diffeomorphic to  $(u, w) \mapsto (-u^2 + w^2, u, \frac{1}{3}uw^4 - \frac{1}{3}w^5)$ , which is diffeomorphic to  $(u, w) \mapsto (u, w^2, w^5)$ . Therefore  $f$  is a  $(2, 5)$ -cuspidal edge.

We conclude this section by posing the problem on singularities of  $\nabla$ -tangent surfaces to torsionless curves in the case of a general affine connection  $\nabla$ .

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